

# ROGERS-RAMANUJAN TYPE IDENTITIES AND NIL-DAHA

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## 0. INTRODUCTION

The theory of Fourier transform of the nilpotent double affine Hecke algebras, *Nil-DAHA*, is applied in the paper to *Rogers- Ramanujan type identities*. Our  $q$ -series are always modular functions (of weight zero) for the congruence subgroups of  $SL(2, \mathbb{Z})$ , which has important implications. There are connections with the algebraic and combinatorial theory of such identities, dilogarithm and *Nahm's conjecture*,  $Y$ -systems, *Demazure characters*, the level-rank duality and *coset algebras*. The latter are associated with tensor products of integrable level-one Kac-Moody representations and are touched only a little in this paper. We mainly study the algebraic and arithmetic aspects.

We arrive at an ample family of the formulas associated with arbitrary (reduced twisted) irreducible affine root systems depending on the choices of initial level-one theta functions (numbered by minuscule weights). Some of our formulas for  $p = 2, 3$  can be identified with known Rogers-Ramanujan identities, but there are new aspects even in these cases. The flexibility with picking the theta functions results in the restricted summations in our identities (for instance, the sums can be even or odd for  $A_1$ ). For  $A_n$ , there are  $\binom{p+n}{n}$  such choices at the level  $p$  (the number of theta functions in the product).

One of the main messages of the paper is that *nil-DAHA can be used to calculate the key string functions for Kac-Moody algebras of type  $A_n$  [KP]* through the expected level-rank duality.

The following topics seem inevitable to put the results of this paper into perspective, but it will be not a systematic review and only basic references will be given. We try to stick here and in the paper mostly to (relatively) known examples; the level-two formulas are the major particular cases we consider. Quite a few topics discussed in the introduction are not touched upon in the main body of this work, so the following is focused on motivation and links to other theories.

**0.1. Using  $q$ -Hermite polynomials.** The reproducing kernel of the Fourier transform of the Nil-DAHA, the *global  $q$ -Whittaker function* from [C4], is actually the key, though it is not introduced and needed in this particular paper. Its explicit expression is equivalent to knowing

$$CT(P_a(X)P_b(X)\theta(X)\mu(X)), \quad \text{where } \mu \stackrel{\text{def}}{=} \prod_{\tilde{\alpha}=[\alpha, j] > 0} (1 - q^j X_\alpha),$$

for all pairs of  $q$ -Hermite polynomials  $P_a$  (their indices  $a, b$  are anti-dominant weights in this paper) and level-one theta functions  $\theta$ . Here the product is over all positive affine roots  $\tilde{\alpha}$ ,  $X_{[\alpha, j]} \stackrel{\text{def}}{=} X_\alpha q^j$ ,  $X_{a+b} = X_a X_b$  for all weights  $a, b$ . By  $CT(\cdot)$ , we mean the constant term of a Laurent series in terms of  $X_a$ .

This is what we really needed. These formulas, used inductively, provide expansions of arbitrary products of level-one theta functions in terms of the  $q$ -Hermite polynomials, which is the essence of this paper.

Then we apply  $CT(\cdot P_c \mu)$  to the expansions of products of theta functions for minuscule weights  $c$ , adjusting them to make the constant term nonzero. It results in the Rogers-Ramanujan type summations, which can be then compared with the formulas for our products in terms of the standard theta functions of level  $p$ , say from Proposition 3.14 from [KP], or with those obtained by direct expansions of such products, or with various formulas in CFT, Conformal Field Theory.

The  $q$ -Hermite polynomials completely disappear from the resulting identities. However their norms, products of  $q$ -factorials in the denominators, and special quadratic forms in the powers of  $q$  in the numerators clearly hint on their presence in the theory.

Interestingly, Rogers' proof of the celebrated Rogers-Ramanujan identities was also based on the  $q$ -Hermite polynomials. The generating function for these polynomials and the formula connecting those for  $q$  and  $q^{-1}$  were the main ingredients. See, e.g. [GIS] for the modern reproduction of his method and some its generalizations. In contrast to his approach, our proof is linked to the global  $q$ -Whittaker function, a kind of generating function for the  $q$ -Hermite polynomials of  $q$ -quadratic type, instead of the usual generating function (cf. [Sus]).

Nil-DAHA can be generally used to manage the standard generating functions (though the formulas are more involved than those for  $A_1$ ), as well as the connection  $q \leftrightarrow q^{-1}$ . Thus the original Rogers method can be potentially extended to general root systems, but we do not discuss it in this work. We note that quite a few multivariable Rogers-Ramanujan type identities in the literature are actually of rank one (for  $A_1$ ) but for higher levels. Here one can (and is supposed to) use the generating function and other special features of the classical rank-one  $q$ -Hermite polynomials at full potential.

Using  $q$ -Hermite polynomials defined for arbitrary root systems seems new in the theory of Rogers-Ramanujan identities. However, let us

mention formula (1.2) from [Ter], where one can see a reduction of our expansion formulas in terms of the  $q$ -Hermite polynomials to a single variable in the case of  $D_n$ . Let us also mention [BCKL]; it is not directly connected with our approach, but the limits of the Macdonald polynomials appear there.

We note that our approach generally results in  $q$ -series where the quadratic forms are multiplied (sometimes divided) by 2 versus the “main stream” of Rogers-Ramanujan type identities, which does not mean of course that they are brand new. For instance, our identities can be identified with known ones for the classical root systems and level two; what is new is their interpretation in terms of the  $q$ -Hermite polynomials and (sometimes) their modular invariance. We note that this is not always clear in what sense known families of the Rogers-Ramanujan type identities are associated with root systems, even if they look as such. For instance, the identities from [An2, War] seem associated with  $B, C$ , but this is not a formal link.

We mention that our procedure is not that smooth in the  $q, t$ -theory based on the Macdonald polynomials instead of the  $q$ -Hermite ones. We still can obtain interesting identities, but the values of the Macdonald polynomials will be present in these identities, and in a significant way. This is unwanted, since not much is known about the meaning of the values of the Macdonald polynomials beyond the evaluation formula and some explicit formulas in lower ranks. Also, the  $q$ -positivity of (all) our formulas and those for the coefficients of  $q$ -Hermite polynomials has no known counterpart in the  $q, t$ -theory.

**0.2. Dual Demazure characters.** The  $q$ -positivity mentioned above is directly related to the geometric meaning of the  $q$ -Hermite polynomials and the interpretation of our construction in terms of coset theory. The second direction is touched upon only a little in this paper; it will be hopefully developed in other works. We certainly cannot interpret at the moment all Rogers-Ramanujan type identities we can obtain via Nil-DAHA in terms of Kac-Moody representation theory, more specifically, via various coset algebras. However, the key step, which is the expansion of the level-one theta function in terms of the  $q$ -Hermite polynomials, can be explained (though this is not a theorem at the moment). It is basically as follows.

Let  $M$  be an irreducible level-one integrable module of the Kac-Moody algebra  $\widehat{\mathfrak{g}}$  in the simply-laced case,  $v_a$  its highest weight vector of weight  $a$  with respect to the Borel subalgebra  $\widehat{\mathfrak{b}}_+$ . By the *dual Demazure filtration* of  $M$ , we mean  $\{\mathcal{F}_b \stackrel{\text{def}}{=} U(\widehat{\mathfrak{b}}_-) v_b\}$  for dominant  $b$  (for  $\widehat{\mathfrak{b}}_+$ ) from the orbit of  $a$  under the action of the (extended) affine Weyl group, where  $v_b$  is the corresponding *extremal vector*  $v_b \in M$ . Consider the corresponding adjoint (graded) module

$$M^{ad} = \oplus_b \mathcal{F}_b^{ad}, \quad \text{where } \mathcal{F}_b^{ad} = \mathcal{F}_b / (+_{c>b} \mathcal{F}_c) \text{ for dominant } b, c$$

in terms of the standard ordering of dominant weights (which are partitions for  $A_n$ ). Then the modules  $\mathcal{F}_b^{ad}$  can be identified with the so-called *global Weyl modules*; see [FeL] and [FoL].

The claim is that the character of  $\mathcal{F}_b^{ad}$  is the character of the corresponding *local Weyl module* upon its multiplication by  $q^{b^2/2}$  and division by the product  $\prod_{i=1}^n \prod_{j=1}^{m_i} (1 - q^j)$ , where  $b = \sum m_i \omega_i$  for fundamental  $\{\omega_i\}$ . Furthermore, the character of the local Weyl module here is the *dual Demazure character* defined as  $D_b^*(X, q) = D_b(X^{-1}, q^{-1})$  for the standard Demazure character  $D_b$ . Here the substitution  $X_a = e^{-a}$  establishes the connection with the standard notation, the character is the trace of  $q^{L_0}$  for the energy operator  $L_0$  from the Virasoro algebra.

The connection of the dual Demazure filtration with the Weyl modules is known in the simply-laced case (see, e.g. [FoL]), but we can not give an exact reference concerning the formula for their characters in terms of the Demazure characters. Though, see formula (3.25) from [FJKMT] in the case of  $A_1$ . Such relation seems not fully established at the moment. We note that using the DAHA-based identities from this paper can be used for the justification of this relation and similar facts. Indeed, generally we know that the formulas in terms of the Demazure operators give characters of modules *no smaller* than the actual ones; then we can use the identities obtained in this paper.

The definition above results in the equality  $D_b^*(X, q) = P_{-b}(X, q)$  for the  $q$ -Hermite polynomials  $P_{-b}$  due to [San] ( $GL$ ) and [Ion] (arbitrary reduced root systems). They established that the level-one Demazure characters in the twisted case are  $P_{-b}(X^{-1}, q^{-1})$  using the DAHA-intertwiners. The substitution  $q \mapsto q^{-1}$  is important here. Changing  $X$  to  $X^{-1}$  is not too significant for dominant weights  $b$ , but this connection holds for any nonsymmetric  $q$ -Hermite polynomials. Let us mention here that the modified Hall-Littlewood polynomials, known

to be connected to Demazure characters in some examples, are closely related to the  $q$ -Hermite polynomials.

**0.3. Toward coset models.** The representation theory interpretation of our identities will be subject of our further research. However, it is important to explain in this paper why we think that the family of Rogers-Ramanujan type identities we obtain is essentially in one-to-one correspondence with the coset decomposition of tensor products of level-one representations, certainly one of the key problems in coset theory.

We refer to [Kac, Kum] for the necessary definitions; also, see paper [FJMT], especially formulas (1.4)-(1.6) there, devoted to the matters closely related to what we discuss below.

Let  $M_1, M_2, \dots, M_p$  be a collection of irreducible integrable representations of  $\widehat{\mathfrak{g}}$  of levels  $l_1, \dots, l_p$ ,  $L = L_{\{\lambda, l\}}$  an irreducible integrable representation of level  $l = l_1 + \dots + l_p$  with the highest weight  $\{\lambda, l\}$  for a (nonaffine) dominant weight  $\lambda$ . We consider the highest weight modules with respect to  $\widehat{\mathfrak{b}}_+$ . Let

$$\nu(L; M_1, \dots, M_p) \stackrel{\text{def}}{=} \text{Hom}_{\widehat{\mathfrak{g}}}(L, M_1 \otimes M_2 \otimes \dots \otimes M_p).$$

This space can be expected to be an irreducible module of the *coset vertex operator algebra* defined essentially as the commutant (centralizer) of  $U(\widehat{\mathfrak{g}})$  diagonally embedded into  $U(\widehat{\mathfrak{g}} \times \dots \times \widehat{\mathfrak{g}})$  ( $p$  times). Importantly, the Virasoro algebra belongs to the coset algebra; using the energy operator  $L_0$  we set

$$\begin{aligned} \chi_q(M_1, \dots, M_p) \text{ and } \chi_q(L; M_1, \dots, M_p) &= \text{trace}(q^{L_0}) \\ \text{for } L_0 \text{ acting in } M_1 \otimes M_2 \otimes \dots \otimes M_p, &\nu(L; M_1, \dots, M_p). \end{aligned}$$

Let  $\widehat{\mathfrak{b}}_-$  be the Borel subalgebra opposite to  $\widehat{\mathfrak{b}}_+$ ,  $\mathfrak{h}$  the Cartan subalgebra and  $\mathbb{C}_{-\lambda}$  the one dimensional  $\widehat{\mathfrak{b}}_-$ -module of weight  $-\lambda$  (for the weight  $\lambda$  above). A standard fact in Kac-Moody theory is that

$$\begin{aligned} \nu(L; M_1, \dots, M_p) &= H_*(\widehat{\mathfrak{b}}_-, \mathfrak{h}; M_1 \otimes M_2 \otimes \dots \otimes M_p \otimes \mathbb{C}_{-\lambda}), \\ &= H_0(\widehat{\mathfrak{b}}_-, \mathfrak{h}; M_1 \otimes M_2 \otimes \dots \otimes M_p \otimes \mathbb{C}_{-\lambda}). \end{aligned}$$

Here  $H_*(\widehat{\mathfrak{b}}_-, \mathfrak{h}; \cdot)$  is relative homology. The higher homology vanishes due to the integrability of the modules we consider; see, e.g. [Kum].

Thus we can identify  $\chi_q(L; M_1, \dots, M_p)$  with the Euler characteristic of the complex

$$\left[ \wedge^*(\widehat{\mathfrak{h}}_-) \otimes M_1 \otimes \dots \otimes M_p \otimes \mathbb{C}_{-\lambda} \right]^\natural,$$

which, in turn, is  $CT(\chi_q(M_1) \cdot \dots \cdot \chi(M_p) \chi_q(\mathbb{C}_{-\lambda}) \mu)$ . When  $c = 0$ , i.e. for the vacuum representation  $L$  of level  $l$ , it will be the constant term of  $\chi_q(M_1) \cdot \dots \cdot \chi(M_p) \mu$ ; the character  $\chi_q(\mathbb{C}_{-\lambda})$  is  $q^{-\lambda}$ , which is  $X_{-\lambda}$  in our notation.

Alternatively, one can express  $M_1 \otimes \dots \otimes M_p$  as a sum of irreducible integrable  $\widehat{\mathfrak{g}}$ -modules and then use the Weyl-Kac character formula for each of them;  $\mu$  is essentially the denominator of this formula.

We see that our identities can be used to determine the characters of the coset algebra acting in the spaces  $\nu(L; M_1, \dots, M_p)$  for level-one  $M_1, \dots, M_p$ . The other way around, one can use these characters (when they are known) to obtain interesting expressions for the Rogers-Ramanujan type series from this paper.

**0.4. Around Nahm's conjecture.** For arbitrary root systems and levels, we arrive at  $q$ -series in the form

$$F_{A,B,C}(q) = \sum_{n \in \mathbb{Z}_+^r} \frac{q^{n^T A n / 2 + B n + C}}{(q_1)_{n_1} \dots (q_r)_{n_r}}, \quad (q)_m = \prod_{i=1}^m (1 - q^i).$$

For  $p = 2$ ,  $A$  is the inverse Cartan matrix ( $r$  is the rank),  $q_i = q^{\nu_i}$ , where  $\nu_i = 1$  for short simple roots  $\alpha_i$  and  $\nu_i = 2, 3$  for long simple  $\alpha_i$  correspondingly for  $B_n$ - $C_n$ - $F_4$  and  $G_2$ . In the simply-laced case, it is exactly the class of series from the so-called *Nahm's conjecture* [Na], generally, for symmetric real positive definite matrices  $A$ .

See [KM, KN, NRT, KKMM] concerning the physics origins of this conjecture, they are not far from our approach (related to the Verlinde algebras). We will not discuss here the key role of the thermodynamic Bethe ansatz (TBA) due to Al. Zamolodchikov and others, the  $Y$ -systems and the cluster algebras; see, e.g. [IKNS, Nak1, Nak2] for recent developments. Also, applications of dilogarithms to volumes of 3-manifolds will be completely omitted.

We note that the method from [NRT, Na] does not rely on TBA and is actually similar to that from [RS] (direct calculating the saddle point). The formula for the sum of  $L(Q_i)$  (the next section) was obtained in [RS] for the inverse Cartan matrix of type  $A_n$  using the asymptotic formula for the partition function  $p(k)$ ; another proof is in [KR].



The simplest cases of our formulas for  $A_1$  and levels  $p = 2, 3$  for the unrestricted theta function (no parity constraints in the summation) can be found in Tables 1, 2 from [Za]; see also Theorems 3.3 and Theorem 3.4 from [VZ]. For  $p = 3$ , we found some new developments.

When the root system is  $A_2$  and  $p = 2$ , the matrix  $A$  is  $\begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{pmatrix}$ . In this case, our formula is also from [Za], [VZ], but we can obtain more identities (generally many more) by using various (level-one) theta functions in the products.

Our matrix  $A$  is associated with the standard bilinear form in tensor product of the weight lattice  $P_R$  (any reduced root system  $R$ , the twisted case) and the root lattice  $Q$  of type  $A_{p-1}$ ;  $p$  is the level. Such class of  $A$  attracts a lot of attention in related physics. The summation can be restricted in our formulas by picking any decomposition of  $p$  as a sum of  $|P_R/Q_R|$  nonnegative terms counting the numbers of theta functions in the  $p$ -product associated with the corresponding minuscule weights. For instance, the number of such choices for  $R = A_n$  equals the number of decompositions  $p = a_1 + \cdots + a_{n+1}$ , where  $a_i \geq 0$  and the order matters; thus it equals  $\binom{n+p}{n}$ .

All our series are modular functions. It allows verifying some conjectural Rogers-Ramanujan type identities and finding new ones. For instance, among other applications, our approach provides a justification of two formulas with question marks in Table 1 from [VZ] (for us, this is the case of  $A_2$ ,  $p = 2$ ). Also we found a split of the formulas from Theorem 3.4 there for  $A = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$ , which relates them to the classical Rogers-Ramanujan identities. It requires using the parity restrictions in the summation, our flexibility with level-one theta functions; this is the case of  $A_1$ ,  $p = 3$ .

We note that the Nahm's conditions hold for  $p = 2$  (all root systems), which was established in [Lee] using  $Y$ -systems; we thank Tomoki Nakanishi for the reference. Our approach, combined with Corollary 3.2 from [VZ] (slightly modified to include the  $BCFG$ -types), gives the rationality of  $\sum_i L(Q_i)$  (see below) for any levels  $p$ , but only for the (unique) distinguished solutions in the range  $\{0 < Q_i < 1\}$ .

To conclude this general discussion (more detail will be provided in the next section), we must note that not many formulas listed in [Za, VZ] and in the vast literature on the Rogers-Ramanujan identities can be obtained by our construction. Recall that only modular

invariant  $q$ -series result from our approach, but certainly not all such. Hopefully, using the root system  $C^\vee C_n$  and  $l \in \mathbb{Z}/2$  (see the end of the paper) will significantly increase the scope of our approach. Also, higher levels  $p$  can be generally used followed by various reductions, but this possibility was not systematically explored at the moment.

**0.5. Dilogarithm identities.** Continuing the previous section, let us briefly discuss using the Rogers dilogarithm

$$L(z) \stackrel{\text{def}}{=} Li_2(z) + \frac{1}{2} \log x \log(1-x),$$

the key in Nahm's conjecture. The modular invariance of the  $q$ -series  $F_{A,B,C}(q)$  above implies the following (see [VZ], [Za] and [Na]).

For a  $N \times N$ -matrix  $A = (a_{ij})$  the system of equations

$$1 - Q_i = \prod_{j=1}^N Q_j^{a_{ij}}, \quad i = 1, \dots, N$$

has a unique solution in the range  $0 < Q_i < 1$ , assuming that  $A$  is real symmetric and positive definite. It is referred to as well-known in physics papers; this system is part of the Thermodynamic Bethe Ansatz (TBA). See Lemma 2.1 from [VZ] for a direct justification.

The main claim is that

$$L_A = \frac{6}{\pi^2} \sum_{i=1}^N L(Q_i) \in \mathbb{Q},$$

assuming the modular invariance of  $F_{A,B,C}(q)$ , which can be interpreted as finding a torsion element in the corresponding Bloch group. See [Za] and [VZ] for the justification. Nahm's conjecture states that any (complex) solutions  $\{Q_i\}$  result in torsion elements in the Bloch group in the same way. Moreover, the latter property is equivalent to the modular invariance of  $F_{A,B,C}(q)$  for suitable  $B, C$ . It appeared generally not the case [VZ], but this conjecture certainly clarifies the role of dilogarithms here.

It is of obvious interest to analyze  $L_A$  for our  $q$ -series and related ones. See [KM] and [KN, NRT, KKMM, Ter] for the  $A$ - $D$ - $E$  cases for  $p = 2$  and papers [IKN, IKNS, Nak1, Nak2] for recent developments. The  $A$ -case ( $p = 2$ ) was calculated in [RS] and [KR] (see also [FrS]).

We note that the tadpole case  $T_n$  corresponds to  $C_n$ , where  $A = (a_{ij})$  for  $a_{i,j} = 2 \min(i, j)$ , which is proportional to the inverse of the

“Cartan matrix” of  $T_n$ . The corresponding  $Q$ -system is *exactly* that for  $A_{2n}$  upon symmetry  $Q_i = Q_{2n-i+1}$  ( $1 \leq i \leq n$ ).

Let us begin with  $A_3, A_4$  and  $D_4$  in the case of level 2. Using the uniqueness of  $\{Q_i\} \subset (0, 1)$ , we can impose the symmetries resulting from those in the corresponding inner products. In these cases, the  $Q$ -systems, their solutions the corresponding  $L = L_A$  are:

$$\begin{aligned}
A_3 : 1 - Q_1 &= Q_1^{\frac{3}{2}} Q_2 Q_3^{\frac{1}{2}}, 1 - Q_2 = Q_1 Q_2^2 Q_3, 1 - Q_3 = Q_1^{\frac{1}{2}} Q_2 Q_3^{\frac{3}{2}}, \\
&\text{setting } Q_1 = Q_3, \quad Q_1 = 2/3 = Q_3, \quad Q_2 = 3/4 \quad \text{and} \quad L = 2; \\
A_4 : 1 - Q_1 &= Q_1^{8/5} Q_2^{6/5} Q_3^{4/5} Q_4^{2/5}, \quad 1 - Q_2 = Q_1^{6/5} Q_2^{12/5} Q_3^{8/5} Q_4^{4/5}, \\
1 - Q_3 &= Q_1^{4/5} Q_2^{8/5} Q_3^{12/5} Q_4^{6/5}, \quad 1 - Q_4 = Q_1^{2/5} Q_2^{4/5} Q_3^{6/5} Q_4^{8/5}, \\
&\text{setting } Q_1 = Q_4, \quad Q_2 = Q_3, \quad Q_1 = 1 - Q_2^{-2} + Q_2^{-1} \quad \text{and} \\
&Q_2 \in (0, 1) \text{ is a unique solution of } t^3 + 2t - t - 1 = 0 : \\
&Q_2 = 2 \cos(\frac{\pi}{7}) - 1 \text{ and } L_{A_4} = \frac{20}{7} \text{ (cf. Watson's identities);} \\
D_4 : 1 - Q_1 &= Q_1^2 Q_2^2 Q_3 Q_4, \quad 1 - Q_2 = Q_1^2 Q_2^4 Q_3^2 Q_4^2, \\
1 - Q_3 &= Q_1 Q_2^2 Q_3^2 Q_4, \quad 1 - Q_4 = Q_1 Q_2^2 Q_3 Q_4^2, \\
&\text{for } Q_1 = Q_3 = Q_4, \quad Q_1 = \frac{3}{4}, \quad Q_2 = \frac{8}{9}, \quad L = 3.
\end{aligned}$$

More generally, our  $L$ -sums are exactly those found in [KM] times  $h/2$  for the Coxeter number  $h = (\rho, \vartheta) + 1$ . Note that our  $Q$ -systems must be transformed to match [KM] following formulas (63,71) there; see formulas (73,77,79,81,83) and (A1) from [Ter] for  $T_n$  (directly related to  $A_{2n}$ ). Namely,

$$\begin{aligned}
L &= \frac{n(n+1)}{(n+3)} \quad \text{for } A_n, \quad Q_n = 2 \cos(\frac{\pi}{n+3}) - 1 \quad \text{when } n = 2m, \\
Q_n &= \frac{1 + \cos(\frac{\pi}{m+2})}{2} \quad \text{for } n = 2m+1 \text{ (see formulas (1,2) in [RS])}; \\
L &= n-1 \quad \text{for } D_n (n > 3), \quad \text{where} \\
Q_i &= \frac{(i+1)^2 - 1}{(i+1)^2} \quad \text{for } i < n-1 \quad \text{and} \quad Q_{n-1} = \frac{n-1}{n} = Q_n; \\
L &= \frac{n(2n+1)}{(2n+3)} \quad \text{for } T_n; \quad L = \frac{36}{7}, \frac{63}{10}, \frac{15}{2} \quad \text{for } E_{6,7,8}.
\end{aligned}$$

The  $Q_i$  are well-known in the  $A_n$ -case; see [KM] or [RS, KR, KN, FrS]. In physics and RT literature,  $2L/h = c_{\text{eff}}$  is called the effective central charge (the finite-size scaling coefficient). It is the difference  $c - 12d_0$  for the corresponding CFT. As such, this is of importance to us, since it provides information on the structure of the right-hand side of the Rogers-Ramanujan type identities we obtain and because we can determine from it which coset models can be expected. See [KM] and the end of this paper for some discussion.

For an arbitrary root system  $R \subset \mathbb{R}^n$ , we normalize the form by the condition  $(\alpha_{\text{short}}, \alpha_{\text{short}}) = 2$ . Then our  $A$  is 2 times the restriction of  $(\cdot, \cdot)$  to the weight lattice. We need to modify the  $Q$ -system and  $L$ :

$$(1 - Q_i)^{\nu_i} = \prod_{j=1}^n Q_j^{a_{ij}} \quad (1 \leq i \leq n), \quad L_R = \frac{6}{\pi^2} \sum_{i=1}^n \nu_i L(Q_i), \quad \nu_i = \frac{(\alpha_i, \alpha_i)}{2}.$$

Then our analysis based on [VZ] results in the following:

$$L_{B_n} = \frac{n(2n-1)}{n+1}, \quad L_{C_n} = n, \quad L_{F_4} = \frac{36}{7}, \quad L_{G_2} = 3.$$

Note that  $L_{B_n} = L_{A_{2n-1}}$ ,  $L_{F_4} = L_{E_6}$ . For  $C_n$ , we have two kinds of dilogarithm identities, this one and that for  $T_n$  (without using  $\nu_i$  on the left-hand side). As a matter of fact, the Rogers-Ramanujan series of type  $R$  in our approach corresponds to the  $Q$ -system for  $R^\vee$ , namely,  $C_n$  corresponds to  $B_n$  and  $B_n$  corresponds to the  $Q$ -system of type  $T_n$ . We will not go into detail.

We are grateful to Tomoki Nakanishi for identifying the formulas for  $L$  in the  $BCFG$  cases with the known instances of the (constant)  $Y$ -systems via the so-called folding construction; see Section 9 from [IKNS] and [Nak1, Nak3]. See also [Nak2] and [IKN], Theorem 2.10.

**0.6. Obtaining “Rogers-Ramanujan”.** Let us demonstrate what our approach gives for the classical Rogers-Ramanujan series, which occur at the level 3 for  $A_1$  in our construction. Recall that the classical interpretation of these identities from [LW] was also associated with  $A_1$  at the level three, though our tools are different and we can connect our formulas with the classical Rogers-Ramanujan identities only upon certain transformations (it actually goes via the coset construction and is less direct than in [LW]).

We begin with the product of three (level-one) theta functions for  $A_1$ , correspondingly even, even and odd, where

$$\theta_k(X) = \sum_{j=-\infty}^{\infty} q^{(2j+k)^2/4} X^j \quad \text{for } k = 0 \text{ (even), } 1 \text{ (odd)}.$$

The  $\mu$ -function is the classical Jacobi theta function :

$$\mu = \prod_{j=0}^{\infty} (1 - X^2 q^j)(1 - X^{-2} q^{j+1}), \quad CT(\mu) = \prod_{j=1}^{\infty} \frac{1}{1 - q^j}.$$

The bilinear form  $CT(f g \mu)$  in the space of symmetric Laurent polynomials in terms of  $X^{\pm 1}$  makes the  $q$ -Hermite polynomials pairwise orthogonal.

The example of  $p = 3$  is related to the Rogers-Ramanujan summations as follows. For  $k = 0, 1$ ,

$$\begin{aligned} \frac{CT(\theta_1 \theta_0 \theta_k (X + X^{-1})^{1-k} \mu)}{q^{\frac{1+k}{4}} \prod_{j=1}^{\infty} (1 - q^j)^2} &= \sum_{n,m \geq 0} \frac{q^{2(n^2 - nm + m^2) + nk + m}}{\prod_{j=1}^{2n+1} (1 - q^j) \prod_{j=1}^{2m+1} (1 - q^j)} \\ &= \sum_{n,m \geq 0} \frac{q^{2(n^2 - nm + m^2) + 2nk - m}}{\prod_{j=1}^{2n+k} (1 - q^j) \prod_{j=1}^{2m} (1 - q^j)} = \sum_{n=0}^{\infty} \frac{q^{2n^2 + 2nk}}{\prod_{j=1}^n (1 - q^{2j})} \prod_{j=1}^{\infty} (1 + q^j)^2. \end{aligned}$$

It is a combination of (3.31, 3.32) with formulas (3.25, 3.28), where we consider the sequences 100, 010 for  $k = 0$  and 101, 110 for  $k = 1$  respectively. Such permutations obviously do not influence the left-hand side, but result in different double summations.

To be more exact, the first two equalities here and their generalizations to arbitrary ranks and levels are the main output of this paper. The reduction to a single summation is special for this particular example. See Section 3.4 for formulas for the remaining combinations of the indices of theta functions and further details.

We arrive at the Rogers-Ramanujan series with  $q^2$  instead of  $q$  :

$$\frac{CT(\theta_1 \theta_0 \theta_k (X + X^{-1})^{1-k} \mu)}{q^{\frac{1+k}{4}} \prod_{j=1}^{\infty} (1 - q^{2j})^2} = \sum_{n=0}^{\infty} \frac{q^{2n^2 + 2nk}}{\prod_{j=1}^n (1 - q^{2j})}.$$

At the end of this paper an outline of the interpretation of this formula for  $k = 1$  in terms of coset theory is provided and we establish a relation of these formulas to the string functions for  $A_2$  of level two.

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## 1. THETA-PRODUCTS VIA NIL-DAHA

Let  $R = \{\alpha\} \subset \mathbb{R}^n$  be a root system of type  $A, B, \dots, F, G$  with respect to a euclidean form  $(z, z')$  on  $\mathbb{R}^n \ni z, z'$ ,  $W$  the *Weyl group* generated by the reflections  $s_\alpha$ ,  $R_+$  the set of positive roots corresponding to fixed simple roots  $\alpha_1, \dots, \alpha_n$ ,  $\Gamma$  the Dynkin diagram with  $\{\alpha_i, 1 \leq i \leq n\}$  as the vertices,  $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$ ,  $R^\vee = \{\alpha^\vee = 2\alpha/(\alpha, \alpha)\}$ .

The root lattice and the weight lattice are:

$$Q = \oplus_{i=1}^n \mathbb{Z}\alpha_i \subset P = \oplus_{i=1}^n \mathbb{Z}\omega_i,$$

where  $\{\omega_i\}$  are fundamental weights:  $(\omega_i, \alpha_j^\vee) = \delta_{ij}$  for the simple coroots  $\alpha_i^\vee$ . Replacing  $\mathbb{Z}$  by  $\mathbb{Z}_\pm = \{k \in \mathbb{Z}, \pm k \geq 0\}$  we obtain  $Q_\pm, P_\pm$ . Here and further see [B].

The form will be normalized by the condition  $(\alpha, \alpha) = 2$  for the *short* roots in this paper. Thus  $\nu_\alpha \stackrel{\text{def}}{=} (\alpha, \alpha)/2$  can be either 1,  $\{1, 2\}$ , or  $\{1, 3\}$ . This normalization leads to the inclusions  $Q \subset Q^\vee, P \subset P^\vee$ , where  $P^\vee$  is defined to be generated by the fundamental coweights  $\{\omega_i^\vee\}$  dual to  $\{\alpha_i\}$ .

We note that  $Q^\vee = P$  for  $C_n (n \geq 2)$ ,  $P \subset Q^\vee$  for  $B_{2n}$  and  $P \cap Q^\vee = Q$  for  $B_{2n+1}$ ; the index  $[Q^\vee : P]$  is  $2^{n-2}$  for any  $B_n$  (in the sense of lattices).

**1.1. Affine Weyl groups.** The vectors  $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \mathbb{R}^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$  for  $\alpha \in R, j \in \mathbb{Z}$  form the *affine root system*  $\tilde{R} \supset R$ ; this is the so-called twisted case.

The vectors  $z \in \mathbb{R}^n$  are identified with  $[z, 0]$ . We add  $\alpha_0 \stackrel{\text{def}}{=} [-\vartheta, 1]$  to the simple roots for the *maximal short root*  $\vartheta \in R_+$ . It is also the *maximal positive coroot* because of the choice of normalization. The Coxeter number is then  $h = (\rho, \vartheta) + 1$ . The corresponding set  $\tilde{R}_+$  of positive roots equals  $R_+ \cup \{[\alpha, \nu_\alpha j], \alpha \in R, j > 0\}$ .

We complete the Dynkin diagram  $\Gamma$  of  $R$  by  $\alpha_0$  (by  $-\vartheta$ , to be more exact); this is called the *affine Dynkin diagram*  $\tilde{\Gamma}$ . One can obtain it from the completed Dynkin diagram from [B] for the *dual system*  $R^\vee$  by reversing all arrows.

The set of the indices of the images of  $\alpha_0$  by all the automorphisms of  $\tilde{\Gamma}$  will be denoted by  $O$ ;  $O = \{0\}$  for  $E_8, F_4, G_2$ . Let  $O' \stackrel{\text{def}}{=} \{r \in O, r \neq 0\}$ . The elements  $\omega_r$  for  $r \in O'$  are the so-called minuscule weights:  $(\omega_r, \alpha^\vee) \leq 1$  for  $\alpha \in R_+$  (here  $(\omega_r, \vartheta) \leq 1$  is sufficient).

**Extended Weyl groups.** Given  $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R}$ ,  $b \in P$ , the corresponding reflection in  $\mathbb{R}^{n+1}$  is defined by the formula

$$(1.1) \quad s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - (z, \alpha^\vee)\tilde{\alpha}, \quad b'(\tilde{z}) = [z, \zeta - (z, b)],$$

where  $\tilde{z} = [z, \zeta] \in \mathbb{R}^{n+1}$ .

The *affine Weyl group*  $\tilde{W}$  is generated by all  $s_{\tilde{\alpha}}$  (we write  $\tilde{W} = \langle s_{\tilde{\alpha}}, \tilde{\alpha} \in \tilde{R}_+ \rangle$ ). One can take the simple reflections  $s_i = s_{\alpha_i}$  ( $0 \leq i \leq n$ ) as its generators and introduce the corresponding notion of the length. This group is the semidirect product  $W \ltimes Q'$  of its subgroups  $W = \langle s_\alpha, \alpha \in R_+ \rangle$  and  $Q' = \{a', a \in Q\}$ , where

$$(1.2) \quad \alpha' = s_\alpha s_{[\alpha, \nu_\alpha]} = s_{[-\alpha, \nu_\alpha]} s_\alpha \quad \text{for } \alpha \in R.$$

The *extended Weyl group*  $\widehat{W}$  generated by  $W$  and  $P'$  (instead of  $Q'$ ) is isomorphic to  $W \ltimes P'$ :

$$(1.3) \quad (wb')([z, \zeta]) = [w(z), \zeta - (z, b)] \quad \text{for } w \in W, b \in B.$$

From now on  $b$  and  $b'$ ,  $P$  and  $P'$  will be identified.

Given  $b \in P_+$ , let  $w_0^b$  be the longest element in the subgroup  $W_0^b \subset W$  of the elements preserving  $b$ . This subgroup is generated by simple reflections. We set

$$(1.4) \quad u_b = w_0 w_0^b \in W, \quad \pi_b = b(u_b)^{-1} \in \widehat{W}, \quad u_i = u_{\omega_i}, \pi_i = \pi_{\omega_i},$$

where  $w_0$  is the longest element in  $W$ ,  $1 \leq i \leq n$ .

The elements  $\pi_r \stackrel{\text{def}}{=} \pi_{\omega_r}$ ,  $r \in O'$  and  $\pi_0 = \text{id}$  leave  $\tilde{\Gamma}$  invariant and form a group denoted by  $\Pi$ , which is isomorphic to  $P/Q$  by the natural projection  $\{\omega_r \mapsto \pi_r\}$ . As to  $\{u_r\}$ , they preserve the set  $\{-\vartheta, \alpha_i, i > 0\}$ . The relations  $\pi_r(\alpha_0) = \alpha_r = (u_r)^{-1}(-\vartheta)$  distinguish the indices  $r \in O'$ . Moreover,

$$(1.5) \quad \widehat{W} = \Pi \ltimes \tilde{W}, \quad \text{where } \pi_r s_i \pi_r^{-1} = s_j \quad \text{if } \pi_r(\alpha_i) = \alpha_j, \quad 0 \leq j \leq n.$$

**The length.** Setting  $\widehat{w} = \pi_r \widetilde{w} \in \widehat{W}$ ,  $\pi_r \in \Pi$ ,  $\widetilde{w} \in \widetilde{W}$ , the length  $l(\widehat{w})$  is by definition the length of the reduced decomposition  $\widetilde{w} = s_{i_1} \dots s_{i_2} s_{i_1}$  in terms of the simple reflections  $s_i$ ,  $0 \leq i \leq n$ . The number of  $s_i$  in this decomposition such that  $\nu_i = \nu$  is denoted by  $l_\nu(\widehat{w})$ .

The length can be also defined as the cardinality  $|\lambda(\widehat{w})|$  of the  $\lambda$ -set of  $\widehat{w}$ :

$$(1.6) \quad \lambda(\widehat{w}) \stackrel{\text{def}}{=} \widetilde{R}_+ \cap \widehat{w}^{-1}(\widetilde{R}_-) = \{\tilde{\alpha} \in \widetilde{R}_+, \widehat{w}(\tilde{\alpha}) \in \widetilde{R}_-\}, \widehat{w} \in \widehat{W}.$$

Alternatively,

$$(1.7) \quad \lambda(\widehat{w}) = \cup_\nu \lambda_\nu(\widehat{w}), \lambda_\nu(\widehat{w}) \stackrel{\text{def}}{=} \{\tilde{\alpha} \in \lambda(\widehat{w}), \nu(\tilde{\alpha}) = \nu\}.$$

See, e.g. [B, Hu] and also [C1, C3].

**1.2. Nil-DAHA.** For pairwise commutative  $X_1, \dots, X_n$ , let

$$(1.8) \quad X_{\tilde{b}} = \prod_{i=1}^n X_i^{l_i} q^k \text{ if } \tilde{b} = [b, k], \widehat{w}(X_{\tilde{b}}) = X_{\widehat{w}(\tilde{b})}.$$

where  $b = \sum_{i=1}^n l_i \omega_i \in P$ ,  $j \in \mathbb{Q}$ ,  $\widehat{w} \in \widehat{W}$ .

For instance,  $X_0 \stackrel{\text{def}}{=} X_{\alpha_0} = qX_{\vartheta}^{-1}$ . We will set  $(\tilde{b}, \tilde{c}) = (b, c)$ , ignoring the affine extensions in this pairing.

Note that  $\pi_r^{-1}$  is  $\pi_{r^*}$  and  $u_r^{-1}$  is  $u_{r^*}$  for  $r^* \in O$ , where the reflection  $*$  is induced by an involution of the nonaffine Dynkin diagram  $\Gamma$ . By  $m$ , we denote the least natural number such that  $(P, P) = (1/m)\mathbb{Z}$ . Thus  $m = 2$  for  $D_{2k}$ ,  $m = 1$  for  $B_{2k}$  and  $C_k$ , otherwise  $m = |\Pi|$ .

**Definition 1.1.** The nil-DAHA  $\mathcal{H}$  is generated over  $\mathbb{Z}_q \stackrel{\text{def}}{=} \mathbb{Z}[q^{\pm 1/m}]$  by the elements  $\{T_i, 0 \leq i \leq n\}$ , pairwise commutative  $\{X_b, b \in P\}$  satisfying (1.8), and the group  $\Pi$ , where the following relations are imposed:

- (o)  $T_i(T_i + 1) = 0$ ,  $0 \leq i \leq n$ ;
- (i)  $T_i T_j T_i \dots = T_j T_i T_j \dots$ ,  $m_{ij}$  factors on each side;
- (ii)  $\pi_r T_i \pi_r^{-1} = T_j$  if  $\pi_r(\alpha_i) = \alpha_j$ ;
- (iii)  $T_i X_b = X_b X_{\alpha_i}^{-1}(T_i + 1)$  if  $(b, \alpha_i^\vee) = 1$ ,  $0 \leq i \leq n$ ;
- (iv)  $T_i X_b = X_b T_i$  if  $(b, \alpha_i^\vee) = 0$  for  $0 \leq i \leq n$ ;
- (v)  $\pi_r X_b \pi_r^{-1} = X_{\pi_r(b)} = X_{u_r^{-1}(b)} q^{(\omega_{r^*}, b)}$ ,  $r \in O'$ .



**T-elements.** Note that one can rewrite (iii,iv) as in [L]:

$$(1.9) \quad T_i X_b - X_{s_i(b)} T_i = \frac{X_{s_i(b)} - X_b}{1 - X_{\alpha_i}}, \quad 0 \leq i \leq n.$$

Given  $\tilde{w} \in \widetilde{W}$ ,  $r \in O$ , the product

$$(1.10) \quad T_{\pi_r, \tilde{w}} \stackrel{\text{def}}{=} \pi_r \prod_{k=1}^l T_{i_k}, \quad \text{where } \tilde{w} = \prod_{k=1}^l s_{i_k}, l = l(\tilde{w}),$$

does not depend on the choice of the reduced decomposition (because  $T_i$  satisfy the same “braid” relations as  $s_i$  do). Moreover,

$$(1.11) \quad T_{\hat{v}} T_{\hat{w}} = T_{\hat{v}\hat{w}} \quad \text{whenever } l(\hat{v}\hat{w}) = l(\hat{v}) + l(\hat{w}) \quad \text{for } \hat{v}, \hat{w} \in \widehat{W}.$$

**Tau-plus.** The following map can be uniquely extended to an automorphism of  $\mathcal{H}$  (see [C1],[C3]):

$$(1.12) \quad \tau_+ : X_b \mapsto X_b, \quad \pi_r \mapsto q^{-\frac{(\omega_r, \omega_r)}{2}} X_r \pi_r, \quad \tau_+ : T_0 \mapsto X_0^{-1}(T_0 + 1),$$

This automorphism fixes  $T_i$  ( $i \geq 1$ ) and  $q^{-1/(2m)}$ , where the latter fractional power of  $q$  must be added to the definition of  $\mathcal{H}$ .

**1.3. Polynomial representation.** The *Demazure operators* are defined as follows:

$$(1.13) \quad T_i = (1 - X_{\alpha_i})^{-1}(s_i - 1), \quad 0 \leq i \leq n;$$

they obviously preserve  $\mathbb{Z}[q][X_b, b \in P]$ . We note that only the formula for  $T_0$  involves  $q$ :

$$(1.14) \quad T_0 = (1 - X_0)^{-1}(s_0 - 1), \quad \text{where} \\ X_0 = qX_{\vartheta}^{-1}, \quad s_0(X_b) = X_b X_{\vartheta}^{-(b, \vartheta)} q^{(b, \vartheta)}, \quad \alpha_0 = [-\vartheta, 1].$$

The map sending  $T_j$  to the corresponding operator from (1.13),  $X_b$  to the operator of multiplication by  $X_b$  (see (1.8)), and  $\pi_r$  ( $r \in O$ ) to  $\pi_r$  induces a  $\mathbb{Z}_q$ -linear homomorphism from  $\mathcal{H}$  to the algebra of linear endomorphisms of  $\mathbb{Z}_q[X]$ . It will be called the *polynomial representation*; the notation is

$$\mathcal{V} \stackrel{\text{def}}{=} \mathbb{Z}_q[X_b, b \in P].$$

It is faithful if  $q$  is not a root of unity.

The polynomial representation is actually the  $\mathcal{H}$ -module induced from the one-dimensional representation  $T_i \mapsto 0$ ,  $\pi_r \mapsto 1$ ,  $r \in O$  of the affine Nil-Hecke subalgebra  $\mathcal{H} = \langle T_i, \pi_r \rangle$ .

**Intertwiners.** Let  $T'_i \stackrel{\text{def}}{=} T_i + 1$ . Given  $\widehat{w} \in \widehat{W}$ , the element  $T'_{\widehat{w}} = \pi_r T'_{i_l} \cdots T'_{i_1}$  does not depend on the choice of the reduced decomposition  $\widehat{w} = \pi_r s_{i_l} \cdots s_{i_1}$ .

We set  $\Psi'_{\widehat{w}} = \tau_+(\pi_r T'_{i_l} \cdots T'_{i_1})$ . Then  $\Psi'_i \stackrel{\text{def}}{=} \tau_+(T'_{-\omega_i})$  for  $i = 1, \dots, n$  are pairwise commutative and, importantly,  $W$ -invariant in the polynomial representation.

Indeed,  $\Psi'_b = \prod_{i=1}^n (\Psi'_i)^{n_i}$  for  $P_- \ni b = -\sum n_i \omega_i$ . Provided that all  $n_i > 0$ , the *reduced* decomposition  $b = b_- = w_0 \pi_{b_+}$  holds for the longest element  $w_0 \in W$  and  $b_+ = w_0(b) \in B_+$ . Thus  $\Psi'_b$  is divisible on the left by  $T'_i = T_i + 1$  for any  $i > 0$  and therefore is divisible on the left by the  $W$ -symmetrizer. It results in the  $W$ -invariance of  $P_b$  for any  $b \in P_-$ .

**$q$ -Hermite polynomials.** The most constructive way to define the  $q$ -Hermite polynomials is by using the intertwiners:

$$(1.15) \quad P_b \stackrel{\text{def}}{=} q^{(b,b)/2} \Psi'_b(1) \quad \text{for } b \in B_-.$$

Here  $\Psi'_i$  can be replaced by their restrictions to  $\mathcal{V}^W$ ; they will become then pairwise commutative  $W$ -invariant difference operators. It is also the most suitable way in this particular paper due to the direct connection with the level-one Demazure characters.

It is important that the expansions of the  $q$ -Hermite polynomials in terms of  $X_b$  and  $q$  have only nonnegative (integral) coefficients. This is the same for their nonsymmetric generalizations. One can deduce the positivity from the interpretation of these polynomials via the Demazure characters or by direct using the DAHA-intertwiners.

As  $q \rightarrow 0$ , the polynomials  $P_b$  become the classical finite dimensional Lie characters, which can be seen, for instance, from (1.19) below.

**1.4. Inner products.** Let  $\mu_o \stackrel{\text{def}}{=} \mu / \langle \mu \rangle$  for

$$(1.16) \quad \mu = \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} (1 - X_{\alpha} q_{\alpha}^j) (1 - X_{\alpha}^{-1} q_{\alpha}^{j+1})$$

and the constant term functional  $\langle \cdot \rangle$  (the coefficient of  $\prod X_i^0$ ). The following is well-known:

$$(1.17) \quad \langle \mu \rangle = \prod_{i=1}^n \prod_{j=1}^{\infty} \frac{1}{1 - q_i^j}, \quad \text{where } q_i = q^{\nu_i}, \quad \nu_i = \nu_{\alpha_i} = (\alpha_i, \alpha_i)/2.$$

The polynomials  $P_b (b \in P_-)$  can be uniquely determined from the conditions:

$$(1.18) \quad P_b - \sum_{c \in W(b)} X_c \in \oplus_{c \succ b} \mathbb{Z}_q X_c \quad \text{and} \quad \langle P_b X_c \mu \rangle = 0 \quad \text{for } c \succ b,$$

$$\{c \succ b\} \stackrel{\text{def}}{=} \{c \in W(b') \mid b' = b + a \in P_- \text{ for } 0 \neq a \in Q_+\}.$$

For  $b, c \in P_-$ , the norm formula reads as:

$$(1.19) \quad \langle P_b P_{c^\iota} \mu_o \rangle = \delta_{bc} \prod_{i=1}^n \prod_{j=1}^{-(\alpha_i^\vee, b)} (1 - q_i^j) \quad \text{for } c^\iota \stackrel{\text{def}}{=} -w_0(c).$$

**Gauss-type inner products.** Let us denote by  $\xi$  the natural projection  $P \rightarrow P/Q$  and fix a nonempty subset  $\varpi \subset P/Q$ . If  $\xi(b) \in \varpi$  then  $\xi(c) \in \varpi$  for all monomials  $X_c$  in  $P_b$ . We will use the symbol  $\mathfrak{tt}$  for the whole  $P/Q$ .

We set:

$$(1.20) \quad \theta_\varpi(X) \stackrel{\text{def}}{=} \sum_{\xi(b) \in \varpi} q^{(b, b)/2} X_b, \quad b \in P.$$

Due to [C2] and [C4], we obtain the following formulas ( $b, c \in P_-$ ):

$$(1.21) \quad \langle \theta_\varpi \mu \rangle = 1, \quad \langle \theta_\varpi \mu_o \rangle = \prod_{i=1}^n \prod_{j=1}^{\infty} (1 - q_i^j),$$

provided that  $0 \in \varpi$  and 0 otherwise,

$$(1.22) \quad \langle P_b(X) P_{c^\iota}(X) \theta_\varpi \mu_o \rangle = q^{\frac{(b-c)^2}{2}} \langle \theta_\varpi \mu_o \rangle$$

for  $\xi(c - b) \in \varpi$  and 0 otherwise.

Recall that  $c^\iota = -w_0(c)$ .

Compare with the “free” formulas:

$$(1.23) \quad \langle \theta_\varpi \rangle = 1 \quad \text{for } 0 \in \varpi \text{ and } 0 \text{ otherwise,}$$

$$\langle X_b X_{-c} \theta_\varpi \rangle = q^{\frac{(b-c)^2}{2}} \langle \theta_\varpi \rangle \quad \text{for } b, c \in P$$

when  $\xi(c - b) \in \varpi$  and 0 otherwise.

Notice that  $b, c$  here are in the whole  $P$ . Switching to nonsymmetric  $q$ -Hermite polynomials in (1.22), provides better matching the free formulas. However, there will be still some differences in the structure of these formulas.

**1.5. Main results.** The following theorem directly results from formulas (1.19, 1.22). We use the following very basic fact from linear algebra and functional analysis.

Provided the convergence, an arbitrary Laurent series  $f(X)$  can be expressed as  $\sum_b (\langle f, e'_b \rangle / \langle e_b, e'_b \rangle) e_b$  for any two bases  $\{e_b\}, \{e'_b\}$  in the space of Laurent polynomials/series orthogonal to each other with respect to a certain nondegenerate form  $\langle \cdot, \cdot \rangle$ .

We consider expressions below as series in terms of nonnegative powers of  $q$  (maybe fractional); analytically, one can assume that  $|q| < 1$ . Recall that  $q_i = q^{\nu_i}$  for  $\nu_i = (\alpha_i, \alpha_i)/2$ .

**Theorem 1.2.** *Let us fix an arbitrary sequence of nonempty subsets  $\varpi = \{\varpi_i \in P/Q, 1 \leq i \leq p\}$  and set  $\theta_i = \theta_{\varpi_i}$ . Then for the sequences  $\mathbf{b} = \{b_i \in P_-, 1 \leq i \leq p\}$ :*

$$(1.24) \quad \langle \mu \rangle^p \prod_{i=1}^p \theta_i = \sum_{\mathbf{b}} \frac{q^{(b_1^2 + (b_1 - b_2)^2 + \dots + (b_{p-1} - b_p)^2)/2}}{\prod_{i=1}^p \prod_{j=1}^n \prod_{k=1}^{-(\alpha_j^\vee, b_i)} (1 - q_j^k)} P_{b_p}(X),$$

*subject to  $\xi(b_1) \in \varpi_1, \xi(b_i - b_{i-1}) \in \varpi_i$  for  $1 < i \leq p$ .*

*In particular, for a fixed  $b_p \in P_-$ , the corresponding sum on the right-hand side of (1.24) depends only on the unordered set  $\{\varpi_i\}$ .  $\square$*

The case of  $p = 1$  reads:

$$(1.25) \quad \langle \mu \rangle \theta_\varpi = \sum_{b \in P_-} \frac{q^{b^2/2}}{\prod_{j=1}^n \prod_{k=1}^{-(\alpha_j^\vee, b)} (1 - q_j^k)} P_b(X) \quad \text{for } \xi(b) \in \varpi.$$

Its iterations based on identities (1.19) and (1.22) readily provide the general (any  $p$ ) formula.

The “free” version of (1.24) based on (1.23) is as follows:

$$(1.26) \quad \prod_{i=1}^p \theta_i = \sum_{\mathbf{b} = \{b_1, \dots, b_p\}} q^{(b_1^2 + (b_1 - b_2)^2 + \dots + (b_{p-1} - b_p)^2)/2} X_{b_p},$$

when  $\xi(b_1) \in \varpi_1, \xi(b_i - b_{i-1}) \in \varpi_i$  for  $1 < i \leq p$ .

The summation here is over all  $b_1, \dots, b_p \in P$  (not only anti-dominant).

We will begin with the key application, which is eliminating the  $q$ -Hermite polynomials by applying  $\langle \cdot, P_{c^t \mu_o} \rangle$  in (1.24).

**Taking the constant term.**

**Corollary 1.3.** *For a given  $c \in P_-$ , assuming that the summation variables  $b_i$  are from  $P_-$  and for an arbitrary sequence of nonempty subsets  $\varpi = \{\varpi_i \subset P/Q, 1 \leq i \leq p\}$ , which determine  $\theta_i = \theta_{\varpi_i}$ ,*

$$(1.27) \quad \Xi_{\varpi}^{p,c} \stackrel{\text{def}}{=} \langle \mu \rangle^p \left\langle \prod_{i=1}^p \theta_i P_{c^i} \mu_o \right\rangle = \sum_{b_1, \dots, b_{p-1}} \frac{q^{(b_1^2 + (b_1 - b_2)^2 + \dots + (b_{p-1} - c)^2)/2}}{\prod_{i=1}^{p-1} \prod_{j=1}^n \prod_{k=1}^{-(\alpha_j^\vee, b_i)} (1 - q_j^k)},$$

$\xi(b_1) \in \varpi_1, \xi(b_i - b_{i-1}) \in \varpi_i \text{ for } 1 < i < p, \xi(c - b_{p-1}) \in \varpi_p.$

Given an arbitrary collection  $\varpi$  of **atomic**  $\varpi_i = \tilde{k}_i \stackrel{\text{def}}{=} \{k_i\}$  for  $k_i \in O \cong P/Q$ , there exists a unique  $c = -\omega_r$  for  $r \in O$  (can be zero) such that using  $P_{c^i}$  makes the left-hand side of (1.27) nonzero, namely, satisfying the congruence  $\omega_r = \sum_{i=1}^n \omega_{k_i} \pmod{Q}$ . Using the notation  $\Xi_{\varpi}^{p,r} = \Xi_{\varpi}^{p,c}$  for  $c = -\omega_r$ , one has:

$$\Xi_{\varpi}^{p,r} = q^{\frac{1}{2} \sum_{i=1}^{p-1} \omega_{k_i}^2 + (\omega_{k_1} + \dots + \omega_{k_{p-1}} - \omega_r)^2/2} \left(1 + \sum_{j=1}^{\infty} C_j q^j\right) \text{ for } C_j \in \mathbb{Z}_+,$$

where  $\varpi = \{\tilde{k}_i\}$  is atomic and  $r$  is defined as above.  $\square$

Alternatively, the product  $\prod_{i=1}^p \theta_i$  on the left hand-side of (1.27) can be calculated using (1.26) and  $\mu_o$  can be replaced by its Laurent expansion, which can be readily obtained directly using the formulas for the action of the lattice  $Q$  on  $\mu$  by translations. Then we will arrive at a  $q$ -series without the denominators. Multiplying it by  $\langle \mu \rangle^p$  and comparing with the right-hand side of (1.27), we will obtain Rogers-Ramanujan type identities. One can also use here a developed machinery of finding formulas for the products of theta functions in terms of the standard ones, more conceptually, the relations in the algebra of theta functions for the powers of an elliptic curve.

## 2. APPLICATIONS, MODULAR INVARIANCE

It is important to understand the actual range of our approach and find generalizations. Accordingly, we consider here not only the formulas that can be obtained by our method, but also neighboring ones, those beyond our reach by now.

**2.1. Level-two formulas for B,C.** The general structure of the multi-dimensional summations as in (1.27) is of course not new. According

to [An2], the existence of multi-dimensional generalizations of the classical Rogers-Ramanujan identities is common in the vast theory of the Rogers-Ramanujan type identities.

The case of  $A_n$  and  $\mathfrak{gl}$  will be examined below. The structure of our level-two formulas for the root systems  $B_n, C_n$  is similar to that of formulas (1.3), (1.8) from [An2], but there are differences.

Let us first consider  $B_n$ . In the notation from [B], our quadratic form is as follows:  $(b, b) = 2 \sum_{i=1}^n u_i^2$  for  $b = \sum_{i=1}^n u_i \varepsilon_i$ . Also,

$$(b, \alpha_j^\vee) = (b, (\varepsilon_j - \varepsilon_{j+1})/2) = u_j - u_{j+1} \text{ for } j < n, (b, \alpha_n^\vee) = (b, \varepsilon_n) = 2u_n.$$

Recall that we normalize the inner product by the “twisted” condition  $(\alpha_{\text{sht}}, \alpha_{\text{sht}}) = 2$  and  $\alpha_{\text{sht}}^\vee = \alpha_{\text{sht}}$ ; also  $q_{\text{sht}} = q$ ,  $q_{\text{lng}} = q^2$ .

Let us take  $c = 0$  and  $\varpi_1 = \{0\} = \varpi_2$ . Setting  $v_i = -u_{n-i+1}$ , one obtains:

$$\langle \mu \rangle^2 \langle \theta_1 \theta_2 \mu_o \rangle = \sum_{0 \leq v_1 \leq v_2 \leq \dots \leq v_n} \frac{q^{2 \sum_{i=1}^n v_i^2}}{\prod_{k=1}^{2v_1} (1 - q^k) \prod_{i=1}^{n-1} \prod_{k=1}^{v_{i+1} - v_i} (1 - q^{2k})}.$$

Compare with the summations in (1.3) and (1.8) from [An2], where correspondingly  $d = 1, 2$ :

$$\sum_{0 \leq v_1 \leq v_2 \leq \dots \leq v_n} \frac{q^{\sum_{i=1}^n v_i^2}}{\prod_{k=1}^{dv_1} (1 - q^k) \prod_{i=1}^{n-1} \prod_{k=1}^{v_{i+1} - v_i} (1 - q^k)}.$$

Let us now consider the root system  $C_n$ . The quadratic form becomes  $(b, b) = \sum_{i=1}^n u_i^2$  for  $b = \sum_{i=1}^n u_i \varepsilon_i$ . One has:

$$(b, \alpha_j^\vee) = (b, (\varepsilon_j - \varepsilon_{j+1})) = u_i - u_{j+1} \text{ for } j < n, (b, \alpha_n^\vee) = (b, \varepsilon_n) = u_n.$$

Taking  $c = 0$ ,  $\varpi_1 = \mathfrak{tt} = \varpi_2$  and setting  $v_i = -u_{n-i+1}$  as above:

$$\langle \mu \rangle^2 \langle \theta_1 \theta_2 \mu_o \rangle = \sum_{0 \leq v_1 \leq v_2 \leq \dots \leq v_n} \frac{q^{\sum_{i=1}^n v_i^2}}{\prod_{k=1}^{v_1} (1 - q^{2k}) \prod_{i=1}^{n-1} \prod_{k=1}^{v_{i+1} - v_i} (1 - q^k)}.$$

The quadratic form here is as in [An2], but there is no match of the denominators.

It is instructional to consider the cases  $B_1$  and  $C_1$ . The corresponding  $B_1$ -series is precisely the  $A_1$ -series from (3.19) for  $\check{\theta}$  (for the lattice  $Q$ ). The  $C_1$ -series becomes that for  $\theta$  there upon substitution  $q^2 \mapsto q$ . So this is exactly as one can expect.

**Product formulas.** Thanks to the Ole Warnaar, we can provide here the product expressions for the  $q$ -series above. In the  $B$ -case, he

used the Bailey pair  $F(1)$ :

$$(2.1) \quad \sum_{0 \leq v_1 \leq v_2 \leq \dots \leq v_n} \frac{q^{2 \sum_{i=1}^n v_i^2}}{\prod_{k=1}^{2v_1} (1 - q^k) \prod_{i=1}^{n-1} \prod_{k=1}^{v_{i+1}-v_i} (1 - q^{2k})} \\ = \frac{(-q^{2n+1}, -q^{2n+3}, q^{4n+4}; q^{4n+4})_\infty}{(q^2; q^2)_\infty}$$

in the notation from (3.10) below. It generalizes (3.19), where  $n = 1$ .

The density of the terms in the numerator versus those in the denominator can be readily calculated. It is  $\frac{6}{4n+6} = \frac{3}{2n+3}$ , where  $1 - \frac{3}{2n+3} = \frac{2n}{2n+3}$ , the density of the missing terms in the numerators, is supposed to be connected with  $c_{\text{eff}}$  for a dual root system. It really coincides with that for  $T_n$  in the introduction or in the table in Section 3.6 (without  $\flat$ ). Actually this is a rigorous mathematical connection, which can be justified following [Za, VZ], but we do not discuss this here in detail.

The  $C_n$ -case is due to David Bressoud (we thank Ole Warnaar for identifying it):

$$(2.2) \quad \sum_{0 \leq v_1 \leq v_2 \leq \dots \leq v_n} \frac{q^{\sum_{i=1}^n v_i^2}}{\prod_{k=1}^{v_1} (1 - q^{2k}) \prod_{i=1}^{n-1} \prod_{k=1}^{v_{i+1}-v_i} (1 - q^k)} \\ = \frac{(q^{n+1}, q^{n+1}, q^{2n+2}; q^{2n+2})_\infty}{(q)_\infty}.$$

Here the density of the terms in the numerator versus the denominator is obviously  $3/(2n+2)$ . Accordingly,  $1 - \frac{3}{2n+2} = \frac{1}{2} \frac{2n-1}{n+1}$  is  $\frac{1}{2}$  times  $c_{\text{eff}}$  for  $B_n$  from Section 3.6. The factor  $\frac{1}{2}$  will disappear if the density is recalculated to the base  $q^2$  as in the previous formula; we will skip the details.

**Warnaar's identities.** Very close formulas to our  $B, C$ -ones can be found in [War]. The formula right after (5.6) there states that:

$$(2.3) \quad \sum_{0 \leq v_1 \leq v_2 \leq \dots \leq v_n} \frac{q^{\sum_{i=1}^n v_i^2}}{\prod_{k=1}^{2v_1} (1 - q^k) \prod_{i=1}^{n-1} \prod_{k=1}^{v_{i+1}-v_i} (1 - q^{2k})} = \prod_{j=1}^{\infty} \frac{1}{1 - q^j},$$

where  $j \not\equiv 2 \pmod{4}$  and  $j \not\equiv 0, \pm 4(n+1) \pmod{8n+12}$ .

Here and in (2.4), the denominators exactly match our ones, but the quadratic forms are divided by 2 versus our formulas. The next but one formula after (5.6) has the quadratic form extended by certain linear terms, which is analogous to our using  $\varpi_1 = \{1\} = \varpi_2$ .

The density of the *missing* terms in the numerator of (2.3) versus those in the denominator is  $\frac{3}{2} \frac{n+1}{2n+3}$ , which is  $\frac{3}{2}(c_{\text{eff}}^b)_{C_n}$  for the (expected) formulas for  $c_{\text{eff}}^b$  from the table in Section 3.6. We will not comment on the factor  $\frac{3}{2}$ .

A counterpart of our  $C$ -case is his (5.14), which reads:

$$(2.4) \quad \sum_{0 \leq v_1 \leq v_2 \leq \dots \leq v_n} \frac{q^{(\sum_{i=1}^n v_i^2)/2}}{\prod_{k=1}^{v_1} (1 - q^{2k}) \prod_{i=1}^{n-1} \prod_{k=1}^{v_{i+1}-v_i} (1 - q^k)} \\ = \frac{(q^{n/2+1/2}, q^{n/2+1}, q^{n+3/2}; q^{n+3/2})_{\infty}}{(-q)_{\infty} (q^{1/2}; q^{1/2})_{\infty}}.$$

The density of the missing terms in the numerator is  $\frac{3}{4} \frac{2n-1}{2n+3}$ , which is  $\frac{3}{4}(c_{\text{eff}}^b)_{B_n}$ , where  $\frac{3}{4}$  is  $\frac{3}{2}$  above divided by 2 due to the base  $q^{1/2}$  in this case, similar to the analysis of (2.2).

There is a natural question. Are identities (1.3) and (1.8) from [An2] and those from [War] somehow associated with root systems  $B, C$ ?

We are very thankful to Ole Warnaar for letting us know about (2.3) and (2.4). We would like to mention Section 5 from [War] with interesting applications to certain Virasoro characters (which may be related to our paper).

**2.2. Weyl algebra and Gaussian sums.** For an integer  $N \geq 1$ , we set  $\zeta = e^{\frac{2\pi i}{N}}$  and pick  $\zeta^{1/(2m)} = e^{\frac{\pi i}{Nm}}$ , where  $(P, P) = \mathbb{Z}/m$  as above. Actually  $\zeta^{1/(2m)}$  can be any  $(2m)$ -th root of  $\zeta$ , not necessarily a primitive  $2mN$ -th root (unless  $2m$  divides  $N$ ). Moreover, such nonprimitive roots do appear and are necessary for the exact analysis of the modularity; this is related to Theorem A from [KP]. In this section we stick to the choice above; accordingly, the modular invariance condition we obtain will be not always sharp.

**The extended Weyl algebra.** This algebra will be denoted by  $\mathcal{W}_N$ . It is generated over  $\mathbb{Q}[\zeta^{1/m}]$  by  $U_a, V_b$  for  $a, b \in P$  and  $w \in W$  subject to the relations,  $U_{a+b} = U_a U_b$ ,  $V_{a+b} = V_a V_b$ ,

$$w U_a w^{-1} = U_{w(a)}, \quad w V_a w^{-1} = V_{w(a)}, \quad U_a V_b U_a^{-1} V_b^{-1} = \zeta^{(a,b)}.$$

Setting  $P[N] \stackrel{\text{def}}{=} P \cap NQ^{\vee}$ , the quotient

$$(2.5) \quad \mathcal{A}_N \stackrel{\text{def}}{=} \mathbb{Q}[\zeta^{1/m}][X_b, b \in P] / (X_c - 1, c \in P[N])$$



has a natural structure of a  $\mathcal{W}_N$ -module:

$$U_b(X_a) = X_{a+b}, \quad V_b(X_a) = \zeta^{-(a,b)} X_a \quad \text{for } a, b \in P.$$

It can be canonically identified with the algebra

$$\text{Funct}(P/P[N]) = \oplus_{\bar{a}} \mathbb{Q}[\zeta] \delta_{\bar{a}}, \quad \text{where } a \in P, \quad \bar{a} = a \bmod P[N].$$

One has:  $\delta_{\bar{a}} \delta_{\bar{b}} = \delta_{\bar{a}, \bar{b}}$  (the latter is the Kronecker delta),

$$U_b(\delta_{\bar{a}}) = \zeta^{(b,a)} \delta_{\bar{a}}, \quad V_b(\delta_{\bar{a}}) = \delta_{\overline{a+b}}, \quad \text{for } a, b \in P.$$

Here and below see Section 3.11.1, Section 3.10.4 from [C3] in the special case  $k_{\text{lng}} = 1 = k_{\text{sht}}$ , and also Lemma 3.11.3 there. We will constantly use that  $(\alpha^\vee, \alpha^\vee) = 1$  for long  $\alpha \in R$ ; the lattice  $Q$  is even.

**Lemma 2.1.** *The coincidence  $P[N] = NQ$  does not hold when and only when*

- (a)  $N$  is odd for the systems  $C_n (n \geq 2)$ ,  $G_2(3|N)$ ,
- (b)  $N$  is even for the systems  $B_n$ ,  $C_n (n \geq 2)$ ,  $F_4$ ,

Moreover,  $P[N] = NQ^\vee$  if  $\nu_{\text{lng}} \mid N$  or  $N \in 2\mathbb{Z}_+$ . Also,  $P[N] = NP$  if  $(N, \nu_{\text{lng}}) = 1$  for  $C_n, F_4, G_2, B_{n \in 2\mathbb{Z}}$ . Here  $(\cdot, \cdot) = \gcd(\cdot, \cdot)$  and by  $a \mid b$ , we mean that  $a$  divides  $b$ .  $\square$

The module  $\mathcal{A}_N$  has a natural *projective* action of  $SL(2, \mathbb{Z})$  that commutes with the action of  $w \in W$  and induces the standard action of  $SL(2, \mathbb{Z})$  on the generators  $\{U_a, V_b\}$ . Namely, for an arbitrary  $\xi \in \mathbb{C}^*$ , we set

$$(2.6) \quad T_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rightsquigarrow \tau_+(\delta_{\bar{a}}) = \frac{1}{\xi} \zeta^{\frac{a^2}{2}} \delta_{\bar{a}} (a \in P),$$

$$(2.7) \quad T_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \rightsquigarrow \tau_-(X_a) = \xi \zeta^{-\frac{a^2}{2}} X_a (a \in P).$$

To use these formulas without problems in  $\mathcal{A}_N$ , we need to check that  $(b + Na)^2 - b^2 = 2N(b, a) + N^2(a)^2$  is divisible by  $2N$  for  $a \in Q^\vee, Na \in P$ . It obviously holds when  $a \in Q$  (the latter is an even lattice) and for even  $N$ . In the  $C$ - $G$  case (a) from Lemma 2.1 (odd  $N$ ,  $3 \mid N$  for  $G_2$ ), we apply (2.6) to the following set of generators for  $P/P[N] = P/NP(C_n)$  and  $P/P[N] = Q/NQ^\vee(G_2)$ :

$$(2.8) \quad P \ni b = \sum_{i=1}^n c_i \omega_i, \quad 0 \leq c_i < N \quad \text{for } C_n (n \geq 2),$$

$$Q \ni b = c_1 \alpha_1 + c_2 \alpha_2, \quad 0 \leq c_1 < N, \quad 0 \leq c_2 < N/3 \quad \text{for } G_2.$$

Let  $\sigma \stackrel{\text{def}}{=} \tau_+ \tau_-^{-1} \tau_+$ . Then one checks that

$$\tau_+ \tau_-^{-1} \tau_+ = \tau_-^{-1} \tau_+ \tau_-^{-1}, \quad \sigma \tau_+^{\pm 1} \sigma^{-1} = \tau_+^{\mp 1}, \quad (\sigma \tau_+^{-1})^3 = \sigma^2.$$

These are the relations of the projective  $PSL(2, \mathbb{Z})$  due to Steinberg (the last two formally follow from the first). Explicitly,

$$\sigma : \delta_{\bar{a}} \mapsto \frac{\gamma}{\xi^3} X_a, \quad X_a \mapsto \frac{\gamma}{\xi^3} \delta_{-\bar{a}} \quad \text{for } \gamma \stackrel{\text{def}}{=} \frac{\sum_{b \in P/P[N]} \zeta^{\frac{b^2}{2}}}{\sqrt{[P : P[N]]}}.$$

In terms of  $\delta_{\bar{a}}$ , the formula for  $\sigma$  becomes as follows:

$$(2.9) \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightsquigarrow \sigma(\delta_{\bar{a}}) = \frac{\gamma}{\xi^3} \sum_b \zeta^{(a,b)} \delta_{\bar{b}} \quad \text{for } a, b \in P/P[N].$$

**Gaussian sums.** We will not discuss here the calculation of the Gaussian sums and the corresponding  $\gamma$ , which are always roots of unity. For  $A_1$ , one obtains that  $\gamma = e^{\frac{\pi i}{4}}$ , the most involved instance of the celebrated four Gauss formulas. We note that the formulas for Gaussian sums can be deduced from the  $q$ -Mehta-Macdonald identities; see [C3], Sections 2.1 and 3.10. Let us provide the list of  $\gamma$  for all root systems. They influence the conductors of the modular functions under consideration if we need to know the exact invariance properties, not up to a character, which is addressed in Theorem 2.3 below.

**Proposition 2.2.** (i) Setting  $\varrho \stackrel{\text{def}}{=} e^{\frac{\pi i}{4}}$ ,

$$(2.10) \quad \begin{aligned} \gamma &= \varrho^n \quad \text{for } A_n, D_n, E_{n=6,7,8}, \\ F_4 : \gamma &= (-1)^{N-1}, \quad G_2 : \gamma = \iota, 1, -1 \quad \text{for } N \bmod 3 = 0, 1, 2, \\ C_n(n \geq 2) : \gamma &= \varrho^{n-\psi}, \quad \text{where } \psi = 0 \quad \text{unless:} \\ &\quad \psi = n \quad \text{for } N = 1 \bmod 4 \quad \text{and} \\ &\quad \psi = 1 \quad \text{for } \{N = 3 \bmod 4 \text{ \& odd } n\}, \\ B_n(n \geq 3) : \gamma &= \varrho^{n-\psi}, \quad \text{where } \psi = 0 \quad \text{unless} \\ &\quad \psi = 4 \quad \text{for } N = 1 \bmod 2. \end{aligned}$$

(ii) In the notation above, let us take  $\xi^3 = \pm \iota \gamma$ , for instance,  $\xi = \pm \varrho = \pm e^{\frac{\pi i}{4}}$  for  $A_1$ . Then the operator  $\sigma^2$  becomes the inversion in  $\text{Funct}(P/P[N])$ , namely,

$$\sigma^2 : \delta_{\bar{a}} \mapsto -\delta_{-\bar{a}}, \quad X_a \mapsto -X_a^{-1}.$$

Accordingly, formulas (2.6) and (2.9) induce the action of the group  $PSL(2, \mathbb{Z})$  in the space  $\mathcal{B}_N \stackrel{\text{def}}{=} \{f \in \mathcal{A}_N \mid w(f) = \text{sgn}(w)z\}$  of the skew-symmetric elements of  $\mathcal{A}_N$ .  $\square$

In the simplest case of  $p = 1$ , i.e. for  $N = h + 1$  the formulas (2.10) for the simply-laced root systems follow from general facts about the Fourier transforms for arbitrary even lattices. See, e.g. paper [VW] for the case of  $\mathfrak{sl}_n$  and [Wu] (esp., Theorem 3.1 and formula (3.9) there). Since, we verified that  $\gamma$  does not depend on  $N$  for  $A$ - $D$ - $E$ , this is sufficient to catch its value.

**2.3. The action of  $SL(2, \mathbb{Z})$ .** By modular invariant functions with respect to a congruence subgroup  $\Gamma \subset SL(2, \mathbb{Z})$  (actually its image in  $PSL(2, \mathbb{Z})$ ), we mean the modular functions (of weight zero) for a certain finite character of  $\Gamma$ . It is up to roots of unity, if an individual  $g \in SL(2, \mathbb{Z})$  is considered. The action of  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$  is standard:  $z \mapsto \frac{az+b}{cz+d}$  for  $q = e^{2\pi iz}$ .

**Theorem 2.3.** *Given level  $p$  as in the theorem, we set  $N = p + h$  for the Coxeter number  $h = (\rho, \vartheta) + 1$  of the root system  $R$  and denote by  $\Gamma^{(N)}$  the kernel of the map  $PSL(2, \mathbb{Z}) \rightarrow \text{Aut}(\mathcal{B}_N)/\mathbb{C}^*$ :*

$$(2.11) \quad T_+ \mapsto \tau_+, \quad T_- \mapsto \tau_-, \quad S \mapsto \sigma \quad \text{for} \quad \xi^3 = \pm i\gamma.$$

For instance, the elements  $\tau_{\pm}^M$  for  $M = 2Nm$  belong to  $\Gamma^{(N)}$ .

Let us fix a minuscule  $c = -\omega_r \in P_-$  or take  $c = 0$  ( $r = 0$ ) and consider all collections  $\varpi = \{\varpi_i \in P/P[1] = P/(P \cap Q^\vee)\}$ . Upon multiplication by a proper common (depending on  $p, R, c$ ) fractional power  $\exp(2\pi iz e/f)$  of  $q = \exp(2\pi iz)$  for  $e, f \in \mathbb{Z}$  ( $f > 0$ ), the  $q$ -series

$$(2.12) \quad \Xi_{\varpi}^{p,r} \stackrel{\text{def}}{=} \left\langle \prod_{i=1}^p \theta_i P_{c^i} \mu_{\circ} \right\rangle \langle \mu \rangle^p$$

from Corollary 1.3 become modular  $\Gamma^{(N)}$ -invariant functions, which are also modular invariant with respect to  $T_+^m$  (up to a character depending on  $c$  and, accordingly, on  $e/f$ ).  $\square$

**On the justification.** It essentially goes as follows. We define the skew-symmetric Looijenga space  $\mathcal{L}_N^{\bar{b}}$  of level  $N = p + h$  for  $\bar{b} \in P/P[N]$

as the linear span of the orbit-sums over  $Q^\vee \rtimes W$ ,

$$\chi_b = \sum_{aw \in Q^\vee \rtimes W} (-1)^{l(aw)} aw(X_b q^{N \frac{x^2}{2}}) \text{ provided that } Na \in P,$$

$$\text{where formally } (aw)(q^{N \frac{x^2}{2}}) = q^{N \frac{(x+a)^2}{2}} = q^{N \frac{a^2}{2}} X_{Na} q^{N \frac{x^2}{2}}.$$

Notice that  $a \in Q^\vee$  here;  $b$  is any pullback of  $\bar{b}$  to  $b \in P$ . Let  $\tilde{\chi}_b$  be  $\chi_b$  divided by the coefficient of  $X_c$  in  $\chi_b$  (if it is nonzero) for  $c$  from (2.12).

The Kac-Moody characters of (twisted) irreducible level- $p$  integrable modules constitute a basis in  $\mathcal{L}_N^{\bar{b}}$  upon their multiplication by  $\mu$  (and the standard eta-type factors in terms of  $q$ ). Concerning the functional equations for the theta functions associated with root systems and the action of  $PSL(2, \mathbb{Z})$  on the Kac-Moody characters and string functions see, e.g, Proposition 3.4 and Theorem A from [KP] and [Kac].

Then we decompose  $\langle \mu \rangle^p (\prod_{i=1}^p \theta_i) P_{c^*} \mu_o$  in terms of such  $\tilde{\chi}_b$ . The modular invariance of the latter under the action of  $g \in PSL(2, \mathbb{Z})$  (up to proportionality) implies that for the coefficients in this decomposition. We use that the coefficients of  $X_c$  were made 1 or zero in  $\tilde{\chi}_b$ . The modular action of  $PSL(2, \mathbb{Z})$  in the skew-symmetric Looijenga space is described in Proposition 2.2 for the skew-symmetric  $W$ -orbit-sums there. It is upon proper normalization, which results in a common factor  $q^{e/f}$  for  $\Xi_{\varpi}^{p,r}$  (given  $c$ , for all  $\varpi$ ).

Since our series are in terms of integral powers of  $q^{1/m}$ , the modular invariance of  $\Xi_{\varpi}^{p,r}$  with respect to  $T_+^m$  is granted (up to roots of unity).

We note that the congruence subgroups of all such  $g$  can be greater than  $\Gamma^{(N)}$  (extended by  $T_+^m$ ) for certain choices of  $R, p$  and  $\varpi$ . Recall that we need to consider the action of  $PSL(2, \mathbb{Z})$  only in the subspace  $\mathcal{B}_N$ . Also, picking a primitive root  $\zeta^{1/(2m)}$  is sufficient but not always necessary. Presumably, taking the (nonprimitive) root  $\zeta^{1/(2m)} = e^{\frac{2\pi i k}{Nm}}$  for odd  $N = 2k - 1$  is always sufficient, but we did not check all details; cf. Theorem A from [KP]. If this is true, then the special treatment of odd  $N$  in the cases of  $C_n$  and  $G_2$  in (2.8) will be unnecessary.

Let us give the simplest example. When  $p = 1$ , the set  $\varpi$  can be only  $\tilde{0} \stackrel{\text{def}}{=} \{0\}$  in the absence of  $P_c$  for minuscule  $c$ ; also  $N = h + 1$  is relatively prime with  $m$  in this case. The subspace of  $W$ -anti-invariant elements in  $\text{Funct}(P/NP)$  is sufficient here instead of  $\text{Funct}(P/P[N])$ . Since this space is one-dimensional, the modular invariance must hold

for the whole  $PSL(2, \mathbb{Z})$ . This analysis is of course not necessary because we know that  $\langle \theta_0 \mu \rangle = 1$ .

We also mention that our series for the greatest  $\varpi = \{\text{ttt}, \dots, \text{ttt}\}$  are generally modular invariant for smaller powers of  $T_-$  than those for generic  $\varpi$ .

**2.4. The A-case.** Let us discuss Corollary 1.3 for  $A_n$ . Then  $\xi : P \rightarrow P/Q = \mathbb{Z}_{n+1}$  and for  $\{b_i\} \subset (P)_-$ ,

$$(2.13) \quad \frac{\langle \prod_{i=1}^p \theta_i P_{c^i} \mu_o \rangle}{(\prod_{j=1}^\infty (1 - q^j))^{pn}} = \sum_{b_1, \dots, b_s} \frac{q^{((b_1)^2 + (b_1 - b_2)^2 + \dots + (b_{p-1} - c)^2)/2}}{\prod_{i=1}^{p-1} \prod_{j=1}^r \prod_{k=1}^{-(\alpha_j, b_i)} (1 - q^k)},$$

$$\xi(b_1) \in \varpi_1, \xi(b_i - b_{i-1}) \in \varpi_i, 1 < i \leq p-1, \xi(c - b_{p-1}) \in \varpi_p.$$

Here  $c = -\omega_r$  ( $r \in O$ ),  $\omega_i^t = \omega_{n-i+1}$ . This correction is necessary to make the summation on the right-hand side nonzero; given  $\varpi$ , such  $c$  is determined uniquely.

**Level-rank duality.** When  $\varpi_i = \text{ttt}$  for all  $i$ , we come to the simplest case of the level-rank duality. Let us divide (2.13) by  $(q)_\infty^{p-1}$  for  $(q)_\infty \stackrel{\text{def}}{=} \prod_{j=1}^\infty (1 - q^j)$ . Then the right-hand side *coincides* with the Rogers-Ramanujan type expression for the  $q$ -character of the module  $L((n+1)\Lambda_0)$  of  $\widehat{\mathfrak{sl}}_p$ . See formula (5.7) from [Geo] and Theorem 7 from [Pr] (and references therein). The origin of this approach is due to [SF].

We note that the division by  $(q)_\infty^{p-1}$  makes perfect sense, since the resulting power  $(q)_\infty^{p(n+1)-1}$  on the left-hand side of (2.13) will become level-rank symmetric, i.e. invariant with respect to  $p \leftrightarrow n+1$ .

We will not discuss in this work the level-rank duality beyond this observation and the case  $n=1, p=3$  considered in Section 3.4.

**Switching to GL.** Let us discuss a  $\mathfrak{gl}_{n+1}$ -version of formula (2.13). The notations are as follows:

$$\mathbb{R}^{n+1} = \oplus_{i=1}^{n+1} \mathbb{R} \varepsilon_i, \quad (b, b) = \sum_{i=1}^{n+1} (u_i)^2 \quad \text{for} \quad b = \sum_{i=1}^{n+1} u_i \varepsilon_i,$$

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (i \leq n), \quad \omega_i = \varepsilon_1 + \dots + \varepsilon_i \quad (i \leq n+1), \quad \omega_0 = 0,$$

$$P = \oplus_{i=1}^{n+1} \mathbb{Z} \varepsilon_i, \quad Q = \oplus_{i=1}^n \mathbb{Z} \alpha_i, \quad P_+ = \{b \in P \mid u_i \geq u_{i+1}\}, \quad P_- = -P_+.$$

We define  $\xi(b) \stackrel{\text{def}}{=} \sum_{i=1}^{n+1} u_i \bmod (n+1) \in \mathbb{Z}_{n+1}$  for  $b \in P$ . Then

$$(2.14) \quad \mu = \prod_{1 \leq i < j \leq n+1} \prod_{k=0}^{\infty} (1 - (Z_i/Z_j)q_{\alpha}^k) (1 - (Z_j/Z_i)q_{\alpha}^{k+1}),$$

$$(2.15) \quad \theta_{\varpi}(X) \stackrel{\text{def}}{=} \sum_{\xi(b) \in \varpi} q^{(b,b)/2} Z_b, \quad b \in P \quad \text{for } \varpi \subset \mathbb{Z}_{n+1}.$$

Here  $b = \sum_{i=1}^{n+1} u_i \varepsilon_i$ ,  $Z_b = \prod_{i=1}^{n+1} Z_i^{u_i}$ ; the constant term will be defined with respect to the variables  $\{Z_i\}$ .

We pick  $\theta_i = \theta_{\varpi_i}$  for a given sequence of subsets  $\varpi = \{\varpi_i \subset \mathbb{Z}_{n+1}\}$ , where  $i = 1, 2, \dots, p$ . Let  $c = -\omega_r$  for  $0 \leq r \leq n$ . Then  $P_{c^*}$  is the  $W$ -orbit-sum of  $(Z_1 \cdots Z_{n-r+1})^{-1}$  for  $r > 0$  and 1 for  $r = 0$ .

Taking the summation elements  $\{b_i\}$  from  $P_-$ , we claim that

$$(2.16) \quad \frac{\langle \prod_{i=1}^p \theta_i P_{c^*} \mu_o \rangle}{\prod_{j=1}^{\infty} (1 - q^j)^{pn}} = \sum_{b_1, \dots, b_{p-1}} \frac{q^{\frac{1}{2}(b_1^2 + (b_1 - b_2)^2 + \dots + (b_{p-2} - b_{p-1})^2 + (b_{p-1} - c)^2)}}{\prod_{i=1}^{p-1} \prod_{j=1}^n \prod_{k=1}^{-(\alpha_j, b_i)} (1 - q^k)},$$

$\xi(b_1) \in \varpi_1, \quad \xi(b_i - b_{i-1}) \in \varpi_i, \quad 1 < i < p, \quad \xi(c - b_{p-1}) \in \varpi_p.$

The left-hand side and therefore the right-hand side do not depend on the order of  $\varpi_i$  in the collection  $\varpi$ .

Let us establish a relation to  $A_n$ . Recall that the fundamental weights for  $A_n$  are connected with those for  $\mathfrak{gl}_n$  as follows:

$$\omega'_i = \sum_{j=1}^i \varepsilon_j - \frac{i}{n+1} \bar{\omega}, \quad \bar{\omega} = \omega_{n+1} = \varepsilon_1 + \dots + \varepsilon_{n+1}, \quad 1 \leq i \leq n.$$

Upon passage to the fundamental weights of  $A_n$ , the connection between formulas (2.16) and (2.13) involves a nontrivial  $\eta$ -type factor, which we are going to calculate.

We use that the square  $(b' + \bar{u}\bar{\omega})^2$  defined for  $\mathfrak{gl}_{n+1}$  is  $(b')^2 + (n+1)\bar{u}^2$  for  $b'$  expressed in terms of  $\omega'_i$  ( $i \leq n$ ). Here  $(n+1)\bar{u} \bmod (n+1)$  equals  $\xi(b)$  for  $b \in P$  and coincides with  $\xi(b')$  defined for  $A_n$ .

Let us consider only the *atomic* sets  $\tilde{k} = \{k\} \subset \mathbb{Z}_{n+1}$  for  $k \in \mathbb{Z}_{n+1}$ . Then the collections  $\varpi = \{\varpi_i, 1 \leq i \leq p\}$  are given by the decompositions  $p = \lambda_0 + \dots + \lambda_n$ , where we treat  $\lambda_k$  as the multiplicity of  $\tilde{k}$  in  $\varpi$ . The formula (2.16) can be presented as follows:

$$\begin{aligned}
(2.17) \quad & \langle \prod_{i=1}^p \theta_i P_{c^\ell} \mu_o \rangle \langle \mu \rangle^p = \prod_{j=0}^n \left( \sum_{s \in j/(n+1) + \mathbb{Z}} q^{s^2/2} \right)^{\lambda_j} \\
& \times \sum_{b'_1, \dots, b'_{p-1}} \frac{q^{((b'_1)^2 + (b'_1 - b'_2)^2 + \dots + (b'_{p-2} - b'_{p-1})^2 + (b'_{p-1} - c')^2)/2}}{\prod_{i=1}^{p-1} \prod_{j=1}^n \prod_{k=1}^{-(\alpha_j, b'_i)} (1 - q^k)}, \text{ where} \\
& \xi(b'_1) \in \varpi_1, \quad \xi(b'_i - b'_{i-1}) \in \varpi_i, \quad 1 < i < p, \quad \xi(c' - b'_{p-1}) \in \varpi_p,
\end{aligned}$$

and the summation is in terms of arbitrary  $\{b'_i\}$  from the lattice  $P'_-$  defined for  $A_n$ . Thus the sum on the right-hand side is exactly as it was in the case of  $A_n$ .

**2.5. Rank 2 level 2.** Let us consider the first nontrivial example in the case of  $A_2$ . See [KKMM] for some coset aspects of this case. We take  $p = 2$  and  $c = 0$ ; then there can be only two admissible atomic collections  $\varpi = \{\varpi_1, \varpi_2\}$  in  $P/Q = \mathbb{Z}_3$ , namely,

$$\odot = \{\tilde{0}, \tilde{0}\}, \quad \otimes = \{\tilde{1}, \tilde{2}\}, \quad \tilde{i} \stackrel{\text{def}}{=} \{i\} \subset \mathbb{Z}_3.$$

The functions  $\Xi_{\varpi}^{2,0}$  for any other collections  $\varpi$  can be linearly expressed in terms of these two. For instance,

$$\Xi_{\tilde{2}, \tilde{1}}^{2,0} = \Xi_{\tilde{1}, \tilde{2}}^{2,0}, \quad \Xi_{\text{tot}} \stackrel{\text{def}}{=} \Xi_{\text{tot}, \text{tot}}^{2,0} = \Xi_{\tilde{0}, \tilde{0}}^{2,0} + 2 \Xi_{\tilde{1}, \tilde{2}}^{2,0}.$$

Thus it suffices to consider the following:

$$(2.18) \quad \Xi_{\odot} = \left\langle \prod_{i=1}^2 \theta_i^2 \mu_o \right\rangle \langle \mu \rangle^2 = \sum_b \frac{q^{b^2}}{\prod_{j=1}^2 \prod_{k=1}^{-(\alpha_j, b)} (1 - q^k)}, \quad b \in P_- \cap Q,$$

$$(2.19) \quad \Xi_{\otimes} = \left\langle \prod_{i=1}^2 \theta_{\tilde{1}} \theta_{\tilde{2}} \mu_o \right\rangle \langle \mu \rangle^2 = \sum_{b \in P_-} \frac{q^{b^2}}{\prod_{j=1}^2 \prod_{k=1}^{-(\alpha_j, b)} (1 - q^k)}, \quad \xi(b) = 1.$$

The sum on the right-hand side of (2.18) plus that in (2.19) times 2, which is  $\Xi_{\text{tot}}$ , can be found in [Za] (Table 2, pg. 47) and in [VZ], Table 1. It is for the matrix  $A = \begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{pmatrix}$  there and for  $B = (0, 0)^{tr}$ . Accordingly,  $b^2 = \frac{2}{3}(v_1^2 + v_1 v_2 + v_2^2)$  for  $b = v_1 \omega_1 + v_2 \omega_2$ . It was expected

in [VZ], based on computer calculations, that

$$(2.20) \quad q^{-1/30} \sum_{b \in P_-} \frac{q^{b^2}}{\prod_{j=1}^2 \prod_{k=1}^{-(\alpha_j, b)} (1 - q^k)} \\ = \frac{1}{\eta(z)} \sum_{n \in \mathbb{Z}} (-1)^n \left( 2q^{\frac{15}{2}(n+\frac{3}{10})^2} + q^{\frac{15}{2}(n+\frac{1}{30})^2} - q^{\frac{15}{2}(n+\frac{11}{30})^2} \right).$$

However this formula was not finally confirmed there. Using  $\Xi_{\text{tt}}$ , we obtain that the left hand-side of (2.20) is a modular function. Then one needs to compare only few terms in the  $q$ -expansions to establish their coincidence. It is a simple computer verification (which we performed).

Let us see what Corollary 2.3 gives in this case. It states that all  $\Xi$  are modular for at least  $\Gamma(30)$  up to a fractional power of  $q$  (which is  $-1/30$ ). Exact formulas for these fractional powers of  $q$  are not discussed in our paper; cf. [Za],[VZ]. Also, the modular invariance of

$$\sharp \Xi \stackrel{\text{def}}{=} q^{-1/30} \Xi, \quad \text{where } \Xi \text{ equals } \Xi_{\odot}, \Xi_{\otimes}, \Xi_{\text{tt}},$$

holds at least with respect to  $T_+^m$  (which is obvious) and for  $T_-^M$  for  $M = 2Nm$ . Recall that  $T_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ .

Since  $N = p + h = 5$ ,  $m = 3$  in this example, the corollary gives that at least  $M = 30$  will be sufficient. However we need to consider only  $\mathcal{B}_N$  (not the whole  $\mathcal{A}_N$ ) in this corollary and also  $\rho \in Q$  for the root system  $A_2$ . Therefore  $M = 15$  is actually sufficient here. This matches our expectation that  $Nm$  instead of  $2Nm$  gives the right (at least more exact) estimate for  $M$  in the case of odd  $N$ .

We note that  $M = 5$  is sharp here for the  $T_-^M$ -invariance of  $\sharp \Xi_{\text{tt}} = q^{-1/30} \Xi_{\text{tt}}$ . However, it is exactly  $M = 15$  for  $\sharp \Xi_{\odot}$  and  $\sharp \Xi_{\otimes}$  (as we computed directly). Recall that by  $\sharp$ , we mean the multiplication by  $q^{-1/30}$ . Moreover,

$$(2.21) \quad (T_-^{-5}) \sharp \vec{\Xi} = \begin{pmatrix} \frac{1}{2} + \frac{i}{\sqrt{12}} & -\frac{4i}{\sqrt{12}} \\ -\frac{2i}{\sqrt{12}} & \frac{1}{2} - \frac{i}{\sqrt{12}} \end{pmatrix} \sharp \vec{\Xi} \quad \text{for } \sharp \vec{\Xi} = \begin{pmatrix} \sharp \Xi_{\odot} \\ \sharp \Xi_{\otimes} \end{pmatrix}.$$

In particular,

$$(2.22) \quad (T_-^{-5}) \sharp \Xi_{\text{tt}} = (T_-^{-5}) (\sharp \Xi_{\odot} + 2 \sharp \Xi_{\otimes}) = \left( \frac{1}{2} - \frac{3i}{\sqrt{12}} \right) \sharp \Xi_{\text{tt}},$$

where  $\frac{1}{2} - \frac{3i}{\sqrt{12}} = e^{-2\pi i/6}$ .



Here we use the expansions of  $\Xi_{\odot}$  and  $\Xi_{\otimes}$ :

$$(2.23) \quad \sharp \Xi_{\odot} = \frac{1}{\eta(z)} \sum_{n \in \mathbb{Z}} (-1)^n \left( q^{\frac{15}{2}(n+\frac{1}{30})^2} - q^{\frac{15}{2}(n+\frac{11}{30})^2} \right),$$

$$(2.24) \quad \sharp \Xi_{\otimes} = \frac{1}{\eta(z)} \sum_{n \in \mathbb{Z}} (-1)^n \left( q^{\frac{15}{2}(n+\frac{3}{10})^2} \right).$$

Using Corollary 1.3 or directly, we observe that

$$\Xi_{\odot, \otimes} = q^{\omega_k^2} \left( 1 + \sum_{j=1}^{\infty} C_j q^j \right),$$

where  $k = 0, 1$  correspondingly. This can be readily checked for the right-hand sides in (2.23–2.24) multiplied by  $q^{1/30}$ ; the smallest powers of  $q$  become respectively  $\frac{1}{30} - \frac{1}{24} + \frac{1}{120} = 0$  and  $\frac{1}{30} - \frac{1}{24} + \frac{27}{40} = \frac{2}{3}$ .

We note that the formulas in Table 1 from [VZ] with nonzero matrix  $B$  (for the same  $A$  as above) correspond to taking nonzero minuscule  $c$  in our construction, i.e. to  $\Xi_{\text{tot}}^{2,r}$  for  $r = 1, 2$ .

**Level-rank duality.** Following Section 3.4 below (in notation from [Geo]), the level-rank duality predicts that

$$\sharp \Xi_{\odot} / (q^{-1/24} \eta(z)) \quad \text{and} \quad \sharp \Xi_{\otimes} / (q^{-1/24} \eta(z))$$

must coincide with the following string functions for  $\widehat{\mathfrak{sl}}_2$  of level 3:

$$q^{-1/120} \frac{\Xi_{\odot}}{\eta(z)} = c_0^{3\widehat{\Lambda}_0}, \quad q^{-1/120} \frac{\Xi_{\otimes}}{\eta(z)} = c_{\alpha_1}^{3\widehat{\Lambda}_0}.$$

The level-rank duality for any  $R, p$  is not discussed in this work.

Indeed, we checked that

$$(2.25) \quad \begin{aligned} & q^{-1/120} (\Xi_{\odot} - \Xi_{\otimes}) / \eta(z) \\ &= q^{-1/120} \sum_{n \in \mathbb{Z}} (-1)^n \left( q^{\frac{15}{2}(n+\frac{1}{30})^2} - q^{\frac{15}{2}(n+\frac{11}{30})^2} - q^{\frac{15}{2}(n+\frac{3}{10})^2} \right) \\ &= \prod_{k=1}^{\infty} (1 - q^{k/3}) \quad \text{where } k \not\equiv \pm 1 \pmod{5}. \end{aligned}$$

The latter product exactly coincides with  $c_0^{3\widehat{\Lambda}_0} - c_{\alpha_1}^{3\widehat{\Lambda}_0} = c_{30}^{30} - c_{12}^{30}$ , where the notation and the formula for the last difference can be found at page 220 in [KP].

## 3. THE RANK-ONE CASE

Here  $\alpha = \alpha_i = \vartheta$ ,  $s = s_1$ ,  $\omega = \omega_1 = \rho$ ; so  $\alpha = 2\omega$  and the standard invariant form reads as  $(n\omega, m\omega) = nm/2$ . Also, we set

$$X = X_\omega = q^x, \quad X(q^{n\omega}) = q^{n/2}, \quad \Gamma(F(X)) = \omega^{-1}(F(X)) = F(q^{1/2}X),$$

i.e.  $x(n\omega) = n/2$ ,  $\Gamma(x) = x + 1/2$ . Also,  $\pi \stackrel{\text{def}}{=} s\Gamma : X \mapsto q^{1/2}X^{-1}$ ;

**3.1. Basic functions.** The  $q$ -Hermite polynomials will be denoted by  $P_n \stackrel{\text{def}}{=} P_{-n\omega} (n \in \mathbb{Z}_+)$  in this section. For instance,  $P_0 = 1$ ,  $P_1 = X + X^{-1}$ ,

$$(3.1) \quad \begin{aligned} P_2 &= X^2 + X^{-2} + 1 + q, \quad P_3 = X^3 + X^{-3} + \frac{1-q^3}{1-q}(X + X^{-1}), \\ P_4 &= X^4 + X^{-4} + \frac{1-q^4}{1-q}(X^2 + X^{-2}) + \frac{(1-q^4)(1-q^3)}{(1-q)(1-q^2)}. \end{aligned}$$

The general formula is classical. For the monomial symmetric functions  $M_0 = 1$ ,  $M_n = X^n + X^{-n}$  ( $n > 1$ ),

$$(3.2) \quad P_n = M_n + \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{(1-q^n) \cdots (1-q^{n-j+1})}{(1-q) \cdots (1-q^j)} M_{n-2j}.$$

These are (continuous)  $q$ -Hermite polynomials introduced by Szegő and considered in many works; see, e.g. [ASI].

Due to (1.15), the composition  $\Psi' = q^{-1/4}(1+T)X\pi$  is the raising operator for the  $P$ -polynomials. Namely, upon its restriction to the symmetric polynomials:

$$(3.3) \quad q^{\frac{n}{2}} \mathcal{R}(P_n) = P_{n+1} \quad \text{for} \quad \mathcal{R} \stackrel{\text{def}}{=} \frac{X^2 \Gamma^{-1} - X^{-2} \Gamma}{X - X^{-1}}.$$

This readily gives (3.2).

The formulas from (1.16), (1.17), (1.19) read as follows:

$$(3.4) \quad \langle P_m(X) P_n(X) \mu_\circ \rangle = \delta_{mn} \prod_{j=1}^n (1 - q^j),$$

where  $m, n = 0, 1, \dots$  and we set  $\mu_\circ = \mu / \langle \mu \rangle$  for

$$(3.5) \quad \mu = \prod_{j=0}^{\infty} (1 - X^2 q^j)(1 - X^{-2} q^{j+1}), \quad \langle \mu \rangle = \prod_{j=1}^{\infty} \frac{1}{1 - q^j}.$$

We note that

$$\omega(\mu) = \mu(Xq^{-1/2}; q) = (-X^2q^{-1})\mu \quad \text{and} \quad \mu(X^{-1}) = -X^{-2}\mu(X).$$

**Theta functions.** We set:

$$(3.6) \quad \begin{aligned} \theta &\stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} q^{n^2/4} X^n \\ &= \prod_{j=1}^{\infty} (1 - q^{j/2})(1 + q^{\frac{2j-1}{4}} X)(1 + q^{\frac{2j-1}{4}} X^{-1}). \end{aligned}$$

Then  $s(\theta) = \theta$  and  $\omega(\theta) = (q^{-1/4}X)\theta$ . We will also use

$$(3.7) \quad \begin{aligned} \check{\theta} &\stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} q^{n^2} X^{2n} \\ &= \prod_{j=1}^{\infty} (1 - q^{2j})(1 + q^{2j-1} X^2)(1 + q^{2j-1} X^{-2}). \end{aligned}$$

Then  $s(\check{\theta}) = \check{\theta}$  and  $\omega(\check{\theta}) = (q^{-1}X^2)\check{\theta}$ . One has (see [ChO]):

$$(3.8) \quad \begin{aligned} \theta\mu &= \sum_{n=0}^{\infty} q^{n(n+2)/12} (X^{n+2} - X^{-n}) \quad \text{for } n \not\equiv 2 \pmod{3}, \\ \langle P_n P_m \theta\mu \rangle &= q^{\frac{(m-n)^2}{4}}, \quad \text{where } n, m \geq 0, \\ \check{\theta}\mu &= \sum_{n=0}^{\infty} q^{n(n+1)/3} (X^{2n+2} - X^{-2n}) \quad \text{for } n \not\equiv 1 \pmod{3}, \\ (3.9) \quad \langle P_n P_m \check{\theta}\mu \rangle &= q^{\frac{(m-n)^2}{4}} \quad \text{for } n - m \in 2\mathbb{Z} \quad \text{and } 0 \text{ otherwise.} \end{aligned}$$

**3.2. Theta-products.** Our general aim is to expand the products of  $\theta$  and  $\check{\theta}$  in terms of the  $P$ -polynomials. We proceed by induction using (3.8, 3.9). Generally,

$$f(X) = \sum_{n=0}^{\infty} \frac{\langle f P_n(X) \mu_{\circ} \rangle}{\langle P_n^2 \mu_{\circ} \rangle} P_n \quad \text{for } \langle P_n^2 \mu_{\circ} \rangle \text{ from (3.4)}$$

and for any  $s$ -invariant Laurent series  $f(X)$ .

Let  $(\mathbf{A}; q)_\infty = \prod_{j=1}^p \prod_{i=0}^\infty (1 - A_j q^i)$  for  $\mathbf{A} = \{A_i, i = 1, \dots, p\}$ . For instance,  $(q; q)_\infty = \prod_{i=1}^\infty (1 - q^i)$ . Accordingly,

$$(3.10) \quad (\mathbf{A}; q)_\mathbf{n} \stackrel{\text{def}}{=} \prod_{i=1}^p \prod_{j=0}^{n_i-1} (1 - A_j q^j) = \prod_{i=1}^p (A_i; q)_{n_i},$$

where  $\mathbf{n} = \{n_i \geq 0, 1 \leq i \leq p\}$ ,  $(a; q)_0 = 1$ . In particular,  $(q)_\mathbf{n} = \prod_{i=1}^p \prod_{j=1}^{n_i} (1 - q^j)$ . All expressions below are considered as series in terms of nonnegative powers of  $q$ ; analytically, one can assume that  $|q| < 1$ .

First of all,

$$(3.11) \quad \frac{\theta(X)}{(q)_\infty} = \sum_{n=0}^\infty q^{\frac{n^2}{4}} \frac{P_n}{(q)_n},$$

$$(3.12) \quad \frac{\check{\theta}(X)}{(q)_\infty} = \sum_{n=0}^\infty q^{n^2} \frac{P_{2n}}{(q)_{2n}}.$$

It will be combined with (3.8, 3.9) in the following particular case of Theorem 1.2.

Let  $\epsilon(a) = a \bmod 2$  for  $a \in \mathbb{Z}$ ,  $\delta_{a,b}^\epsilon = 1$  if  $\epsilon(a) = \epsilon(b)$  and 0 otherwise. For  $\mathbf{n} = \{n_1, \dots, n_p\} \subset \mathbb{Z}_+$ , we set

$$\epsilon_{i,j}^{(\mathbf{n})} = \epsilon(n_i - n_j) \quad \text{for } 0 \leq i, j \leq p, \quad \text{where } n_0 \stackrel{\text{def}}{=} 0.$$

**Theorem 3.1.** *Let us take a sequence  $\boldsymbol{\xi} = \{\xi_i = 0, \text{ or } 1, 1 \leq i \leq p\}$ , setting  $\theta_i = \theta$  if  $\xi_i = 1$  and  $\theta_i = \check{\theta}$  for  $\xi_i = 0$ . Thus  $\varpi = \{0\}$  for  $\xi = 0$  and  $\varpi = 1 = \mathbb{Z}_2$  for  $\xi = 1$ .*

For  $\mathbf{n} = \{n_i \in \mathbb{Z}_+, 1 \leq i \leq p\}$ ,

$$(3.13) \quad \frac{\prod_{i=1}^p \theta_i(X)}{(q)_\infty^p} = \sum_{\mathbf{n}} \frac{q^{\binom{n_1^2 + (n_1 - n_2)^2 + \dots + (n_{p-1} - n_p)^2}{4}}}{(q)_\mathbf{n}} P_{n_p}(X),$$

$$\text{subject to } \epsilon_{i,i-1}^{(\mathbf{n})} = 0 \text{ if } \xi_i = 0 \text{ for } 1 \leq i \leq p; \quad \epsilon_{1,0}^{(\mathbf{n})} = \epsilon(n_1).$$

In particular for any fixed  $n_p \in \mathbb{Z}_+$ , the corresponding subsum on the right-hand side of (3.13) depends only on the number of indices  $i$  such that  $\xi_i = 0$  but not on their specific order in the sequence  $\boldsymbol{\xi}$ .

**Calculating the coefficients.** The simplest example of nontrivial combinatorial identities obtained by permuting  $\{\xi_i\}$  is for  $p = 3$  and

$\xi = \{0, \mathfrak{tt}, \mathfrak{tt}\}$ ,  $\xi' = \{\mathfrak{tt}, 0, \mathfrak{tt}\}$ . Considering the coefficient of  $P_k$  for  $k = 0, 1, \dots$ , we obtain the following identities:

$$\sum_{\mathbf{n}} \delta_{n_1,0}^\epsilon \frac{q^{\binom{n_1^2+(n_1-n_2)^2+(n_2-k)^2}{4}}}{(q)_{n_1,n_2}} = \sum_{\mathbf{n}} \delta_{n_1,n_2}^\epsilon \frac{q^{\binom{n_1^2+(n_1-n_2)^2+(n_2-k)^2}{4}}}{(q)_{n_1,n_2}}.$$

It suffices to assume here that  $n_2$  is odd in the both sides and that  $n_1$  is even on the left-hand side, correspondingly, odd on the right-hand side. As a matter of fact, this holds for any fixed  $n_2$ , which can be deduced from the Euler alternating identity (2.2.1) from [MSZ]; see also [An1], (2.2.6).

Now, taking the constant term of (3.13) and using (3.2), we arrive at the following identities:

$$(3.14) \quad \left\langle \frac{\prod_{i=1}^p \theta_i(X) X^m}{(q)_\infty^p} \right\rangle = \sum_{\mathbf{n}} \frac{q^{\binom{n_1^2+(n_1-n_2)^2+\dots+(n_{p-1}-n_p)^2}{4}}}{(q)_{n_p/2+m/2} (q)_{n_p/2-m/2} \prod_{i=1}^{p-1} (q)_{n_i}}$$

subject to  $\epsilon_{i,i-1}^{(\mathbf{n})} = 0$  if  $\xi_i = 0$  ( $i \geq 0$ ) and  $\epsilon(n_p - m) = 0$ ,

where the indices  $k$  in  $(q)_k$  are assumed nonnegative.

When  $p = 1$ , we readily obtain a well-known identity (see, e.g, [Za], formula (27)):

$$(3.15) \quad \frac{1}{(q)_\infty} = \sum_{k,l \geq 0} \frac{q^{kl}}{(q)_k (q)_l} \text{ subject to } k = l + m \text{ for any } m \in \mathbb{Z}.$$

It is called the Durfee rectangle identity. Its particular case  $m = 0$  (corresponding to  $m = 0$  in (3.14)) is the Euler identity:

$$(3.16) \quad \frac{1}{(q)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n^2}.$$

See [MSZ], formula (2.1.6) and [An1], (2.2.8) (the Cauchy identity).

**3.3. The case  $p=2$ .** For  $\xi_i = \mathfrak{tt}$  ( $i = 1, 2$ ),

$$(3.17) \quad \begin{aligned} \frac{\langle \theta^2 \rangle}{(q)_\infty^2} &= \frac{\sum_{i=-\infty}^{\infty} q^{i^2/2}}{(q)_\infty^2} = \frac{\prod_{j=0}^{\infty} (1 + q^{j+1/2})^2}{(q)_\infty} \\ &= \sum_{\mathbf{n}} \frac{q^{\binom{n_1^2+(n_1-n_2)^2}{4}}}{(q)_{n_1} (q)_{n_2/2}^2}, \text{ where } \epsilon(n_2) = 0. \end{aligned}$$

For an arbitrary  $m \in \mathbb{Z}$ , the coefficient of  $X^m (m \geq 0)$  here reads:

$$(3.18) \quad \frac{\langle X^m \theta^2 \rangle}{(q)_\infty^2} = q^{m^2/2} \frac{\sum_{i=-\infty}^{\infty} q^{i^2/2}}{(q)_\infty^2} \\ = \sum_{\mathbf{n}} \delta_{n_2, m}^\epsilon \frac{q^{(n_1^2 + (n_1 - n_2)^2)/4}}{(q)_{n_1} (q)_{n_2/2 + m/2} (q)_{n_2/2 - m/2}},$$

assuming that  $n_2 \pm m \geq 0$ .

**Constant term with mu.** Following Corollary 1.3, let us now apply  $\langle \cdot \mu_o \rangle$  to (3.13) for  $p = 2$  and, correspondingly, for  $\xi = \{\text{tt}, \text{tt}\}$  and  $\xi = \{0, 0\}$ . One has:

$$(3.19) \quad \frac{\langle \theta^2 \mu_o \rangle}{(q)_\infty^2} = \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q)_n}, \quad \frac{\langle \check{\theta}^2 \mu_o \rangle}{(q)_\infty^2} = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q)_{2n}}.$$

Using (3.8) and formulas (2.8.10) and (2.8.9) from [MSZ],

$$(3.20) \quad \frac{\langle \check{\theta}^2 \mu_o \rangle}{(q)_\infty^2} = \frac{\sum_{j=-\infty}^{\infty} q^{4j^2-j}}{(q^2; q^2)_\infty} = \frac{(-q^3, -q^5, q^8; q^8)_\infty}{(q^2; q^2)_\infty}.$$

For  $\varpi = \{1\}$  (i.e. for  $\xi = 1$ ), we will denote the corresponding  $\theta_\varpi$  by  $\hat{\theta}$ . Then

$$(3.21) \quad \frac{\langle \hat{\theta}^2 \mu_o \rangle}{(q)_\infty^2} = \frac{\langle \theta^2 \mu_o \rangle}{(q)_\infty^2} - \frac{\langle \check{\theta}^2 \mu_o \rangle}{(q)_\infty^2} = q^{1/2} \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q)_{2n+1}} \\ = q^{1/2} \frac{\sum_{j=-\infty}^{\infty} q^{4j^2-3j}}{(q^2; q^2)_\infty} = q^{1/2} \frac{(-q, -q^7, q^8; q^8)_\infty}{(q^2; q^2)_\infty}.$$

**Modular invariance.** The relations (3.21, 3.21) are modulo 8 counterparts of the Rogers-Ramanujan identities, which are modulo 5. They can be found in Table 1 at pg. 44 in [Za]; see also Theorem 3.3 from [VZ]. There are 3 entries there for  $A = 1$ . The first gives that

$$(3.22) \quad \frac{\langle \theta^2 \mu_o \rangle}{(q)_\infty^2} = \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q)_n} = q^{1/48} \eta(z)^2 / (\eta(z/2) \eta(2z)) \\ \text{for } \eta(z) = q^{1/24} \prod_{i=1}^{\infty} (1 - q^i), \text{ where } q = e^{2\pi i z}.$$

The second entry in this table is directly connected with our

$$\frac{\langle \theta^2 \mu_o(X + X^{-1}) \rangle}{(q)_\infty^2} = q^{1/4} \sum_{n=0}^{\infty} \frac{q^{(n^2-n)/2}}{(q)_n} = 2q^{1/4-1/24} \eta(2z)/\eta(z).$$

We see that the function  $f_{\mathbf{ttt}} = q^{-1/48} \frac{\langle \theta^2 \mu_o \rangle}{(q)_\infty^2}$  is modular invariant with respect to  $\Gamma(2)$  extended by  $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . The action of  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  from  $SL(2, \mathbb{Z})$  is  $z \mapsto \frac{az+b}{cz+d}$  and

$$\begin{aligned} \Gamma(M) &= \{A = (a_{ij}) \mid a_{ij} \in \delta_{ij} + M\mathbb{Z}\}, A \in SL(2, \mathbb{Z}), \\ \Gamma_0(M) &= \{A \mid a_{ij} \in \delta_{ij} + M\mathbb{Z} \text{ for } \{ij\} \neq \{21\}\}. \end{aligned}$$

To be more exact,  $f_{\mathbf{ttt}}$  is strictly invariant with respect to  $z \mapsto -1/z$  and  $f_{\mathbf{ttt}}(z+2) = e^{-\frac{\pi i}{12}} f_{\mathbf{ttt}}(z)$ . It is in contrast to

$$f_0 = q^{-1/48} \frac{\langle \check{\theta}^2 \mu_o \rangle}{(q)_\infty^2}, \quad f_1 = q^{1/2-1/48} \frac{\langle \hat{\theta}^2 \mu_o \rangle}{(q)_\infty^2},$$

because these two functions are only  $\Gamma_0(16)$ -invariant (up to a finite character). Moreover,

$$f_0(1/(8z+1)) = e^{2\pi i/6} f_1(z) \quad \text{and} \quad f_1(1/(8z+1)) = e^{2\pi i/6} f_0(z),$$

which matches  $f_{\mathbf{ttt}}(1/(2z+1)) = e^{\frac{\pi i}{12}} f_{\mathbf{ttt}}(z)$  because  $f_{\mathbf{ttt}} = f_0 + f_1$ .

For an arbitrary  $p$  and any collections  $\varpi$  and corresponding minuscule  $c \in P_-$ , Corollary 2.3 provides the following upper bound for the modular invariance of the corresponding series (upon multiplication by a proper fractional powers of  $q$ ); it must be modular invariant at least with respect to the congruence subgroup  $\Gamma_0(4(p+2)) \cap \Gamma(2)$ . Thus this estimate is sharp in the case of  $p = 2$ .

**3.4. The case  $p=3$ .** Here  $N = 5$ . Let us allow minuscule  $c = -\omega_r$  in Corollary 1.3, adding  $P_0 = 1$  or  $P_1 = X + X^{-1}$  to the formula for  $r = 0, 1$ . We will consider only the *atomic* sets  $\tilde{k} = \{k\} \subset \mathbb{Z}_2$  for  $k = 0, 1$ . The admissible choices for  $\varpi = \{\omega_1, \omega_2, \omega_3\}$  are as follows:

$$\begin{aligned} r = 0 : \quad & \mathbf{000} = \{\tilde{0}, \tilde{0}, \tilde{0}\}, \quad \mathbf{110} = \{\tilde{1}, \tilde{1}, \tilde{0}\}, \\ r = 1 : \quad & \mathbf{100} = \{\tilde{1}, \tilde{0}, \tilde{0}\}, \quad \mathbf{111} = \{\tilde{1}, \tilde{1}, \tilde{1}\}. \end{aligned}$$

We will not distinguish the sequences that can be obtained from each other by permutations since they result in coinciding  $\Xi_{\varpi}^{3,r}$ . As we discussed above, there are nontrivial identities reflecting this coincidence.

Recall our main result. Provided that  $u + v + w + r = 0 \pmod{2}$ ,

$$(3.23) \quad \Xi_{uvw}^{3,r} = \frac{\langle \theta_u \theta_v \theta_w P_r \mu_o \rangle}{(\prod_{i=1}^{\infty} (1 - q^i))^3} = \sum_{n_1, n_2 \geq 0} \frac{q^{(n_1^2 - n_1 n_2 + n_2^2 - n_2 r)/2 + r^2/4}}{(q)_{n_1} (q)_{n_2}},$$

subject to  $\epsilon(n_1) = u, \epsilon(n_2) = u + v$  for  $u, v, w, r \in \mathbb{Z}_2$ .

When  $\varpi = \mathfrak{tt}$ , these formulas are from the last entry of Table 1 in [Za] for the matrix  $\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$ . See also the table from Theorem 3.4 from [VZ] (the first entry there). The latter table provides the eta-type formulas for  $\Xi_{\mathfrak{tt}}^{3,0}$  and  $\Xi_{\mathfrak{tt}}^{3,1}$ , but we need them here for atomic  $\varpi$ . They are certain (not quite trivial) splits of the formulas given in [VZ]:

$$(3.24) \quad q^{-1/20} \Xi_{\mathfrak{oo}}^{3,0} = (\theta_{5, \frac{3}{4}}(2z) + \theta_{5, \frac{13}{4}}(2z)) \eta(z) / (\eta(2z) \eta(z/2)) \\ - \theta_{5,2}(2z) \eta(2z) / \eta(z)^2,$$

$$(3.25) \quad q^{-1/20} \Xi_{\mathfrak{io}}^{3,0} = \theta_{5,2}(2z) \eta(2z) / \eta(z)^2, \text{ where}$$

$$(3.26) \quad \theta_{5,m}(z) \stackrel{\text{def}}{=} \sum_n (-1)^{[n/10]} q^{n^2/40} \text{ for } n \in 2m - 1 + 10\mathbb{Z},$$

$$(3.27) \quad q^{-4/20} \Xi_{\mathfrak{iii}}^{3,1} = \theta_{5, \frac{3}{2}}(z) \theta_{5,2}(2z) \eta(z)^3 / (\eta(z/2)^2 \eta(2z)^2) \eta(10z) \\ - \theta_{5,1}(2z) \eta(2z) / \eta(z)^2,$$

$$(3.28) \quad q^{-4/20} \Xi_{\mathfrak{ioo}}^{3,1} = \theta_{5,1}(2z) \eta(2z) / \eta(z)^2.$$

**Level-rank duality.** The summations in the formulas (3.23) naturally appear in the theory of  $\widehat{\mathfrak{sl}}_3$ . Namely, our summations divided by  $\eta(z)^2$  and with proper fractional powers of  $q$  coincide with formulas (5.16-19) in [Geo] for certain level-two *string functions*. As a matter of fact, the string functions listed there are all independent ones for such a level. We note that  $q^{-2/15}$  and  $q^{-1/30}$  in [Geo] and below are our  $q$ -power corrections (necessary to ensure the modular invariance of  $\Xi$  and the string functions) adjusted due to the division by  $\eta(z)^2$ :

$$-\frac{1}{20} = -\frac{2}{15} + \frac{1}{12}, \quad -\frac{4}{20} = -\frac{1}{30} + \frac{1}{12} - \frac{1}{4}.$$



Using the notations from [Geo] (see also [KP]),

$$\begin{aligned} \frac{q^{-2/15+1/12}}{\eta(z)^2} \Xi_{\infty}^{3,0} &= c_0^{2\hat{\Lambda}_0}, & \frac{q^{-1/20-1/12}}{\eta(z)^2} \Xi_{110}^{3,0} &= c_{\alpha_1+\alpha_2}^{2\hat{\Lambda}_0}, \\ \frac{q^{-1/30+1/12-1/4}}{\eta(z)^2} \Xi_{100}^{3,1} &= c_{\hat{\Lambda}_1}^{\hat{\Lambda}_0+\hat{\Lambda}_1}, & \frac{q^{-1/30+1/12-1/4}}{\eta(z)^2} \Xi_{111}^{3,0} &= c_{\hat{\Lambda}_1+\alpha_2}^{\hat{\Lambda}_0+\hat{\Lambda}_1}. \end{aligned}$$

We hope to discuss this duality for arbitrary (atomic)  $\varpi$  in further works.

**Modular invariance.** Let us briefly discuss the modular properties of these functions. It is directly connected with the action of  $PSL(2, \mathbb{Z})$  in the 4-dimensional Verlinde algebra for  $A_1$  of level 3; the corresponding products of  $\theta_{\varpi}$  form a basis in this space. This guarantees that the whole  $PSL(2, \mathbb{Z})$  acts in the linear space generated by  $\Xi_{uvw}^{3,r}$  from (3.23). Explicitly, this action is as follows.

All four functions are modular invariant with respect to  $(T_-)^M$  where  $M = 10$  (exactly) and under the action of  $T_+$  (up to proportionality). The total modular invariance subgroup is  $\Gamma_0(10)$  (up to a character).

As for  $T_-^5$ , setting

$$\begin{aligned} \# \Xi^{\rightarrow 0} &= q^{-1/20} (\Xi_{\infty}^{3,0}, \Xi_{110}^{3,0})^{tr}, \quad \# \Xi^{\rightarrow 1} = q^{-4/20} (\Xi_{111}^{3,1}, \Xi_{100}^{3,1})^{tr}, \\ (3.29) \quad (T_-^5) (\# \Xi^{\rightarrow 0}, \# \Xi^{\rightarrow 1}) &= \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} (\# \Xi^{\rightarrow 0}, \# \Xi^{\rightarrow 1}). \end{aligned}$$

The eigenvalues of  $T_+$  are correspondingly

$$\pm e^{2\pi i/20} \text{ for } \# \Xi_{100}^{3,1}, \# \Xi_{111}^{3,1} \text{ and } \pm e^{-2\pi i/20} \text{ for } \# \Xi_{\infty}^{3,0}, \# \Xi_{110}^{3,0}.$$

This is actually obvious, since the  $q$ -series for  $q^{-1/4} \Xi_{100}^{3,1}$ ,  $q^{1/4} \Xi_{111}^{3,1}$  and  $\Xi_{\infty}^{3,0}$ ,  $q^{-1/2} \Xi_{110}^{3,0}$  contain only integral powers of  $q$ , and  $T_+(q^{1/2}) = -q^{1/2}$ . The corresponding eigenvalues will be only due to the fractional  $q$ -powers in (3.24, 3.25) and (3.27, 3.28), i.e. due to the  $q$ -normalization.

Furthermore,  $T_-^2$  preserves the two-dimensional spaces

$$\mathbb{C} \# \Xi_{100}^{3,1} \oplus \mathbb{C} \# \Xi_{110}^{3,0} \text{ and } \mathbb{C} \# \Xi_{\infty}^{3,0} \oplus \mathbb{C} \# \Xi_{111}^{3,1}.$$

Setting now  $\# \Xi^{\leftarrow} = (\# \Xi^{\leftarrow 0}, \# \Xi^{\leftarrow 1})$  for

$$\# \Xi^{\leftarrow 0} = (q^{-4/20} \Xi_{100}^{3,1}, q^{-1/20} \Xi_{110}^{3,0})^{tr}, \quad \# \Xi^{\leftarrow 1} = (q^{-4/20} \Xi_{111}^{3,1}, q^{-1/20} \Xi_{\infty}^{3,0})^{tr},$$

$$(3.30) \quad (T_-^2)^{\# \Xi} = \begin{pmatrix} \cos(\frac{\pi}{5}) + i \frac{\sin(\pi/5)}{\sqrt{5}} & 2i \frac{\sin(\pi/5)}{\sqrt{5}} \\ 2i \frac{\sin(\pi/5)}{\sqrt{5}} & \cos(\frac{\pi}{5}) - i \frac{\sin(\pi/5)}{\sqrt{5}} \end{pmatrix}^{\# \Xi}.$$

Since the modular transformations  $T_+$ ,  $T_-^5$  and  $T_-^2$  act in this 4-dimensional space, the whole  $PSL(2, \mathbb{Z})$  acts here, and in a sufficiently explicit way.

Some of these facts are well known. The functions  $\Xi_{100}^{3,1}$  and  $\Xi_{110}^{3,0}$  are directly related to the Rogers-Ramanujan identities with  $q^2$  instead of  $q$  and a simple common eta-type factor:

$$(3.31) \quad q^{-\frac{1}{4}} \Xi_{100}^{3,1} = q^{-\frac{1}{20}} \theta_{5,1}(2z) \frac{\eta(2z)}{\eta(z)^2} = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{\prod_{j=1}^n (1 - q^{2j})} \prod_{j=1}^{\infty} (1 + q^j)^2,$$

$$(3.32) \quad q^{-\frac{1}{2}} \Xi_{110}^{3,0} = q^{-\frac{9}{20}} \theta_{5,2}(2z) \frac{\eta(2z)}{\eta(z)^2} = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{\prod_{j=1}^n (1 - q^{2j})} \prod_{j=1}^{\infty} (1 + q^j)^2.$$

Compare with the classical formulas:

$$(3.33) \quad G(z) = \sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = q^{\frac{1}{60}} \frac{\theta_{5,1}(z)}{\eta(z)}, \quad H(z) = \sum_{n \geq 0} \frac{q^{n^2+n}}{(q)_n} = q^{-\frac{11}{60}} \frac{\theta_{5,2}(z)}{\eta(z)}.$$

See, e.g. formula (23) from [Za]. For instance, combining (3.29) and (3.30) we arrive at formula (24) from [Za], describing the action of  $z \mapsto -1/z$  on  $G, H$ .

Let us comment on the powers of  $q$  that appear here. The functions  $q^{-1/40} \theta_{5,1}(z)$  and  $q^{-9/40} \theta_{5,2}(z)$  are power series in terms of integral powers of  $q$ , which is obvious from the definition in (3.26) and can be immediately seen from (3.33); use that  $q^{-1/24} \eta(z)$  is such a series. Upon  $z \mapsto 2z$ , we readily arrive at the fractional  $q$ -powers in (3.31, 3.32).

**3.5. A coset interpretation.** We will focus only on formula (3.32). According to Section 0.3, we need to consider three level-one integrable modules  $M_1 = L_{1,0}$ ,  $M_2 = L_{1,0}$ ,  $M_3 = L_{0,1}$  of  $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$  with the highest weights  $\{1, 0\}$ ,  $\{1, 0\}$  and  $\{0, 1\}$  in the standard notation. The module  $M_3 = L_{0,1}$  is the so-called vacuum representation, where  $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$  acts as zero. Then  $\nu(L; M_1, M_2, M_3) = \text{Hom}_{\widehat{\mathfrak{sl}}_2}(L, M_1 \otimes M_2 \otimes M_3)$  is a natural representation of the coset algebra defined for the diagonal embedding  $\widehat{\mathfrak{sl}}_2 \hookrightarrow \widehat{\mathfrak{sl}}_2 \times \widehat{\mathfrak{sl}}_2 \times \widehat{\mathfrak{sl}}_2$ .

Here we consider  $M_1 \otimes M_2 \otimes M_3$  as a submodule of  $M^{\otimes 3}$  for  $M = L_{1,0} \oplus L_{0,1}$ . The passage to the adjoint (graded) module followed by

taking the constant term with  $\mu$  is standard here. It is more subtle for the identity from (3.31), where we need to multiply the corresponding character by  $P_1$  before taking the constant term.

The level-rank duality says that the coset  $(\widehat{\mathfrak{sl}}_2 \times \widehat{\mathfrak{sl}}_2 \times \widehat{\mathfrak{sl}}_2)/\widehat{\mathfrak{sl}}_2$  (all are of level 1) is the same as the coset of  $\widehat{\mathfrak{sl}}_3$  at the level 2 over the Heisenberg algebra

$$\widehat{\mathfrak{h}}_3 = \mathfrak{h}_3 \otimes \mathbb{C}[t, t^{-1}] \text{ for the Cartan subalgebra } \mathfrak{h}_3 \subset \mathfrak{sl}_3.$$

The coset  $(\widehat{\mathfrak{sl}}_3/\widehat{\mathfrak{h}}_3)$  can be naturally considered as a product of two level-2 cosets  $(\widehat{\mathfrak{sl}}_3/\widehat{\mathfrak{gl}}_2)$  and  $(\widehat{\mathfrak{sl}}_2/\widehat{\mathfrak{h}}_2)$ , where we denote by  $\widehat{\mathfrak{h}}_2$  the Heisenberg algebra in  $\widehat{\mathfrak{sl}}_2$  and use the standard embedding  $\widehat{\mathfrak{gl}}_2 \hookrightarrow \widehat{\mathfrak{sl}}_3$ . This presentation is simply because of consecutive taking the invariants. Since the level is 2, these two cosets belong correspondingly to the Virasoro  $\{4, 5\}$  and  $\{3, 4\}$  minimal models.

The term  $\prod_{j=1}^{\infty} (1 + q^j)$  on the right-hand side of (3.32) corresponds to the Virasoro module of weight  $1/16$  from the  $\{3, 4\}$  minimal model. In the standard notation, it is  $\varphi_{1,2}$ .

The remaining term

$$\prod_{j=1}^{\infty} (1 + q^j) \sum_{n=0}^{\infty} q^{2n^2+2n} / \left( \prod_{j=1}^n (1 - q^{2j}) \right)$$

from (3.32) is of more involved nature. Namely, it is the character of the Virasoro module  $\varphi_{2,2}$  from the minimal model  $\{4, 5\}$ .

For this identification, we use that the Virasoro  $\{4, 5\}$  minimal model is related to the minimal models for the Neveu-Schwarz and Ramond superalgebras. For instance, it gives that the Virasoro module  $\varphi_{2,2}$  can be viewed as an irreducible representation of the Ramond algebra. The latter algebra has a basis  $\{L_i, S_j\}$ , where  $L_i$  are Virasoro generators and  $S_j$  are odd satisfying  $[S_i, S_j]_+ = L_{i+j}$ . Thus knowing the action of  $\{S_j\}$  is sufficient. One can check that  $\varphi_{2,2}$  has a monomial basis

$$\{S_{-1}^{a_1} S_{-2}^{a_2} \cdots S_{-m}^{a_m}\}, \text{ where } 0 \leq a_i < 4 \text{ and}$$

$$\text{if } a_i > 1, \text{ then both } a_{i-1} \leq 1 \text{ and } a_{i+1} \leq 1.$$

This results in the required formula for the character and gives a coset interpretation of (3.32). The remaining three identities from (3.24, 3.27, 3.28) can be also interpreted in this way, which is quite interesting in coset theory.

**3.6. Theta of level  $1/2$ .** According to our analysis, the “main stream” of the theory of Rogers-Ramanujan type identities (for instance, papers [An2, War]) can be connected with “square roots” of level-one theta functions. Stimulated by formulas (2.3) and (2.4), we defined and calculated (numerically) the  $L$ -sums upon division of the  $A$ -matrices considered above by 2. They appeared rational for all root systems (see below). We will denote them by  $L_R^b$ .

It is explained in [Nak3] that these formulas can be obtained from known identities using formula (1.7) from [Nak1] for  $\ell = 3$ ; see Corollary 1.9 there. In the  $BCFGT$ -cases, the so-called folding construction can be used; see Section 9 from [IKNS] and [Nak3]. Importantly, the  $Y$ -systems establish a *direct* link to the Langlands functoriality of the *affine* root systems. We will touch it upon below, when discussing functoriality properties of the effective central charges.

Let  $c_{\text{eff}} = L_R^b/h$  for the Coxeter number  $h$ ; notice that it was  $2L/h$  before. The matrix  $A_R^b = A_R/2 = (a_{ij}^b)$  has the entries  $a_{ij}^b = (\omega_i, \omega_j)$ . Recall that the weight lattice  $P$  is supplied with the standard form  $(\cdot, \cdot)$  normalized by the condition  $(\alpha_{\text{sht}}, \alpha_{\text{sht}}) = 2$ .

In terms of  $\nu_i = \frac{(\alpha_i, \alpha_i)}{2}$ , the  $Q$ -system is as follows:

$$(3.34) \quad (1 - Q_i)^{\nu_i} = \prod_{j=1}^n Q_j^{a'_{ij}} \quad (1 \leq i \leq n), \quad L'_R = \frac{6}{\pi^2} \sum_{i=1}^n \nu_i L(Q_i);$$

$A'$  here is  $A$  or  $A^b$ . The  $Q$ -system in the case of  $T_n$  identically coincides with that for  $A_{2n}$  (with  $^b$  or without) with a reservation that the number of terms in  $L'_{T_n}$  is  $n$  versus  $2n$  for  $L'_{A_{2n}}$ . The  $T$ -type  $Q$ -system is defined for  $C_n$ , but without using  $\nu_i$  in (3.34).

We note that a somewhat different  $Q$ -system naturally appears when applying the method from [VZ] (and previous works):

$$(3.35) \quad 1 - Q_i = \prod_{j=1}^n Q_j^{a'_{ij}/\nu_j} \quad (1 \leq i \leq n), \quad \tilde{L}'_R = \frac{6}{\pi^2} \sum_{i=1}^n \frac{\nu_{\text{Ing}}}{\nu_i} L(Q_i),$$

where  $A'$  is  $A$  or  $A^b$ . It is simple to see that  $\tilde{L}'_R = L'_{R^\vee}$  for  $R = B_n, C_n, F_4, G_2$ .

**The table.** Let us compare the values of  $L$  and the corresponding  $c_{\text{eff}}$  without and with  $^b$ . To avoid misunderstanding with the normalization, let us list the coefficients  $a_{11}^b$ . They are  $n/(n+1)$  for  $A_n$ ,  $4/3$  for  $E_6$ ,

2 for  $B_n, E_7, G_2$  and 1 otherwise. The values of  $L$  and effective central charges are as follows:

$R_n$	$L_R$	$c_{\text{eff}}$	$L_R^b$	$c_{\text{eff}}^b$
$A_n$	$n(n+1)/(n+3)$	$2n/(n+3)$	$n(n+1)/(n+4)$	$n/(n+4)$
$B_n$	$n(2n-1)/(n+1)$	$(2n-1)/(n+1)$	$2n(2n-1)/(2n+3)$	$(2n-1)/(2n+3)$
$C_n$	$n$	$1$	$2n(n+1)/(2n+3)$	$(n+1)/(2n+3)$
$D_n$	$n-1$	$1$	$2(n-1)n/(2n+1)$	$n/(2n+1)$
$E_6$	$36/7$	$6/7$	$24/5$	$2/5$
$E_7$	$63/10$	$7/10$	$6$	$1/3$
$E_8$	$15/2$	$1/2$	$80/11$	$8/33$
$F_4$	$36/7$	$6/7$	$24/5$	$2/5$
$G_2$	$3$	$1$	$8/3$	$4/9$
$T_n$	$n(2n+1)/(2n+3)$	$2n/(2n+3)$	$n(2n+1)/(2n+4)$	$n/(2n+4)$

Table 1. Values of  $L_R, L_R^b$  and  $c_{\text{eff}}, c_{\text{eff}}^b$

See [KM] about the history, physics meaning, merits and demerits of  $c_{\text{eff}}$  in the  $A$ - $D$ - $E$  elastic scattering theories. Presumably it measures the massless degrees of freedom of a theory, somehow analogous to the calculation we performed above in several examples of the “fraction of the missing terms” in the numerator of product formulas. One of the problems is that  $c_{\text{eff}}$  (without  $b$ ) do not reflect well the connections between different root systems. Also, the effective central charges for the series  $D_n$  are not what one could expect. This problem seems to be better addressed when using  $A^b = A/2$ .

Indeed, the coincidences and other relations of the effective central charges in the  $b$ -case become practically always meaningful in Kac-Moody theory and in the theory of coset models. The effective charges are equal to each other for the following pairs:

$$D_{n+1} \leftrightarrow C_n, \quad B_n \leftrightarrow A_{2n-1}, \quad E_6 \leftrightarrow F_4 \leftrightarrow D_2, \quad G_2 \leftrightarrow D_4, \quad E_7 \leftrightarrow A_2.$$

The relation  $2(c_{\text{eff}}^b)_{E_8} = (c_{\text{eff}}^b)_{D_{16}}$  also makes sense from the viewpoint of representation theory. Almost all of these relations are directly related to the symmetries of the  $Q$ -systems (and the uniqueness of the solutions in the range  $\{0 < Q_i < 1\}$ ) and to the Langlands duality for the affine root systems. See [Nak3] and references therein. Moreover, let us mention that the  $A_n$  central charges are generally greater than  $1/2$  and intersect the  $D$ -charges, strictly smaller than  $1/2$ , only at  $D_1$  and  $D_3$ . A similar relation holds for  $B$  and  $C$ .

For the physics part, the breakthrough paper [Zam] (the three-state Potts model,  $c = \frac{4}{5}$ ) and its further developments for the  $W(A_n)$ -algebras result in  $c = 2n/(n+3)$  (see [KM]). It is the central charge of the unitary CFT describing the  $\mathbb{Z}_{n+1}$ -parafermions, which coincides with  $(c_{\text{eff}})_{A_n}$ . In unitary theories,  $d_0 = 0$  so  $c_{\text{eff}} = c$ . In the  $\flat$ -case, the central charge for  $A_n$  is  $n/(n+4)$  (maybe up to a common coefficient of proportionality).

Our approach, based on expanding products of level-one starting theta functions in terms of the  $q$ -Hermite polynomials, cannot be directly extended to the  $\flat$ -case. Theta functions of level  $1/2$  and the corresponding Kac-Moody modules are needed here. The latter modules are not available in integrable Kac-Moody theory, but the Virasoro and  $W$ -algebras, parafermions, Neveu-Schwarz and Ramond superalgebras provide certain substitutes. DAHA seems to have greater flexibility here, though this cannot be translated to Kac-Moody theory (so far).

**Nakanishi's note.** Let us comment on [Nak3]; we thank Tomoki Nakanishi for various discussions. It suggests that one can try to obtain the  $\flat$  case by considering  $p = 3$  (coinciding with  $\ell$  in [Nak3]) combined with the folding construction. Indeed, following his note, the approach via Nahm's conjecture gives the rationality of  $\tilde{L}_R^\flat$ . However, one needs the theory of  $Y$ -systems to obtain these rational numbers. It is challenging that such a reduction (from  $2n$  variables for  $p = 3$  to  $n$  variables using the symmetry of  $A_2$ ) cannot be seen at the level of the corresponding Rogers-Ramanujan identities.

Let us provide some details. For an arbitrary root system  $R$ , taking  $\varpi = \{\text{ttt}, \text{ttt}, \text{ttt}\}$  and  $c = 0$  for  $p = 3$ , the series from (1.27) reads as follows:

$$(3.36) \quad \sum_{b_1, \dots, b_2} \frac{q^{(b_1^2 + (b_1 - b_2)^2 + (b_2)^2)/2}}{\prod_{i=1}^2 \prod_{j=1}^n \prod_{k=1}^{-(\alpha_j^\vee, b_i)} (1 - q_j^k)},$$

where the summation is over all possible  $b_1, b_2 \in P_-$ . If one formally makes here  $b_1 = b = b_2$ , then it becomes

$$(3.37) \quad \sum_b \frac{q^{b^2}}{\prod_{j=1}^n \prod_{k=1}^{-(\alpha_j^\vee, b)} (1 - q_j^k)^2},$$

i.e. with the same  $q$ -powers in the numerators and the squares of the denominators versus the non- $\flat$  case with  $p = 2$ .

We do not see direct relations between the modular properties of (3.37) and (3.36), but one can proceed as follows. The uniqueness Lemma 2.1 from [VZ] implies that the canonical solution of the  $Q$ -system for (3.36) in the range  $\{0 < Q_i < 1\}$  is invariant under the transposition  $b_1 \leftrightarrow b_2$ . Therefore it also solves the  $Q$ -systems for (3.37), which is the  $b$ -ones due to the squares in the denominators of (3.37).

More generally, the  $Q$ -systems can be reduced from  $(p-1)n$  variables to  $[p/2]n$  variables for any given root system  $R$  and level  $p$ ; we impose the symmetry corresponding to the standard automorphism of  $A_{p-1}$ . Let us assume that Nahm's conjecture holds in this situation and that there is a similar reduction for all (complex) solutions of the  $Q$ -system; the uniqueness claim guarantees it only in the range  $\{0 < Q_i < 1\}$ . Then this makes the modular invariance of the Rogers-Ramanujan series in the  $b$ -case for  $p-1$  a (conditional) corollary of that in the non- $b$ -case for  $p$ . We hope to extend the DAHA approach to obtain this fact without using Nahm's conjecture.

In conclusion, we note that the exact rational values of the  $L$ -sums can be generally obtained from the product formulas or similar identities for the modular Rogers-Ramanujan series. DAHA methods can be used here, but we touch the product-type formulas only a little in this paper. Therefore such identities can be viewed as “quantization” of the  $Q$ -systems and the corresponding dilogarithm formulas. Nahm's conjecture outlines the class of  $Q$ -systems that can be “quantized,” i.e. lifted to modular invariant Rogers-Ramanujan type series. Continuing this line, one can try to interpret the coordinate Bethe ansatz, associated with unitary DAHA modules and analytic properties of the eigenfunctions of the corresponding QMBP, as some kind of quantization of TBA and the  $Y$ -systems, but this is beyond the present paper.

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